What is Factorization Homology?

Chris Grossack
(they/them)

March 23, 2024

Why?

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(if you try hard enough)

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Didn't this talk have "homology" in the title?

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What?

Following Lurie's notes[1], we recall one conceptual proof of classical Poincaré Duality goes as follows:

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- We know what $H_\bullet(\mathbb{R}^n, A)$ *must* be, since we can just compute

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- We know how to compute $H_\bullet(M, A)$ by using Mayer-Veitoris, covering M by copies of $\mathbb{R}^n$
- Precisely, one can show that

$$C_\bullet(M, A) \simeq \varinjlim_{\mathbb{R}^n \subseteq M} C_\bullet(\mathbb{R}^n, A)$$

then take homology

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Is there a conceptual algebraic definition of what's happening here?

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- There are *classifying categories* $\mathcal{G}/\mathcal{A}/\mathcal{R}$/etc. so that a group/abelian group/ring/etc. is a finite product preserving functor from the classifying category to Set.

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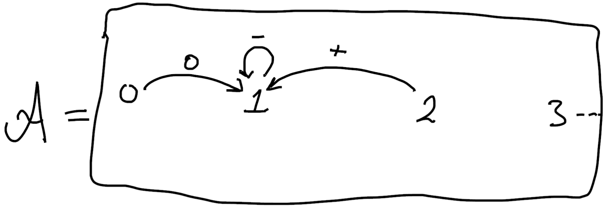
- There are *classifying categories* $\mathcal{G}/\mathcal{A}/\mathcal{R}$/etc. so that a group/abelian group/ring/etc. is a finite product preserving functor from the classifying category to Set.
- We can fruitfully study relationships between algebraic theories by studying the classifying categories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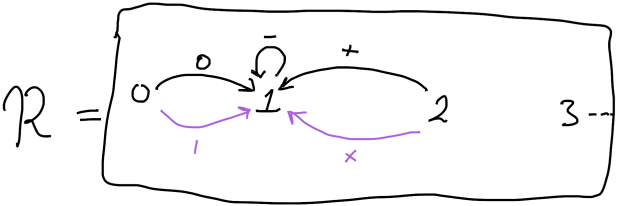
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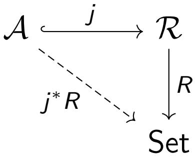

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$$\begin{array}{ccc} \mathcal{A} & \overset{j}{\hookrightarrow} & \mathcal{R} \\ & \searrow^{j^* R} & \downarrow R \\ & & \mathsf{Set} \end{array}$$

This is the usual forgetful functor Ring $\to$ Ab!

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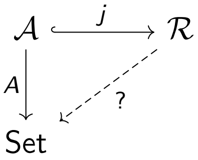
Given an abelian group A, is there a way to *freely extend* our functor to one defined on $\mathcal{R}$?



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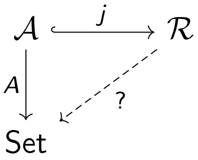

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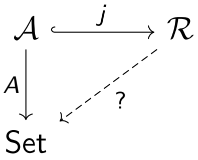

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Moreover, $\mathrm{Lan}_j(A) : \mathcal{R} \to \mathrm{Set}$ is exactly the usual free ring on A

Wasn't this talk supposed to be about factorization homology?

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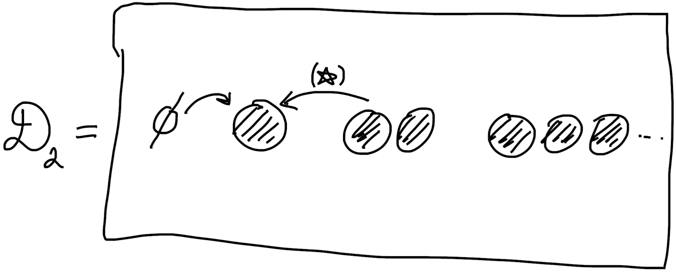
Let's take a second to see what this means when $n = 2$


$\mathcal{D}_2 =$
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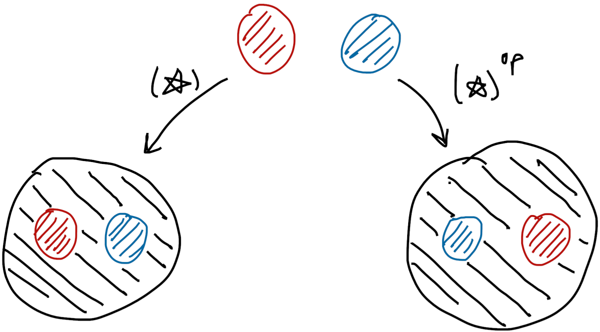
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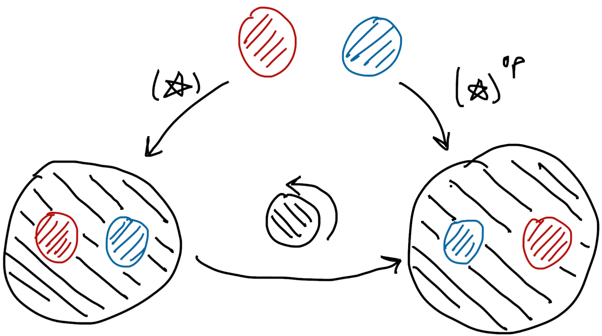
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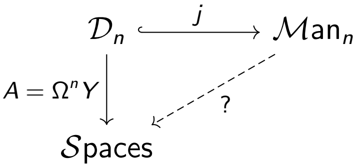

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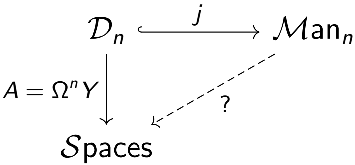

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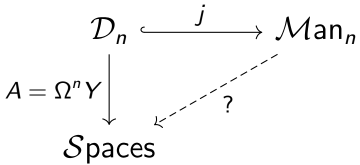

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Analogous to a graded algebra, only the gradings are indexed by (connected) manifolds. Moreover, we have operations between the "graded pieces" corresponding to smooth embeddings.

How?

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The answer is **Collared Excision**!

If $M = M_0 \times \mathbb{R}$ is a *collared manifold*, then the smooth embedding $M \coprod M \to M$ induces an algebra structure $\int_M A \otimes \int_M A \to \int_M A$.

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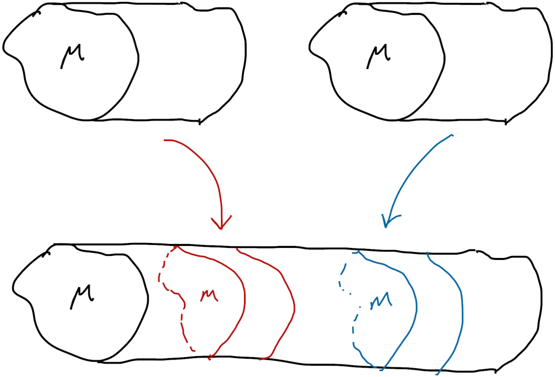

Moreover, if N is a manifold with an M *shaped collar*, in the sense that it has M as part of its boundary, the embedding $N \coprod M \to N$ gives $\int_N A$ a right (resp. left) $\int_M A$-module structure

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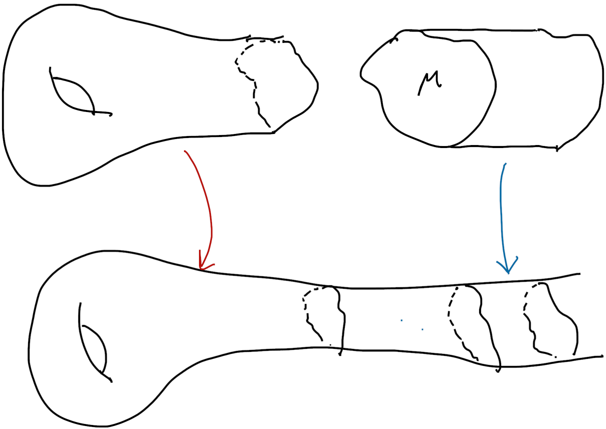

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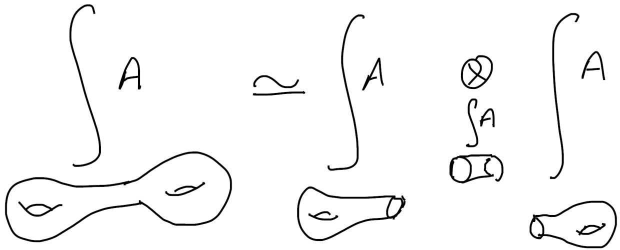

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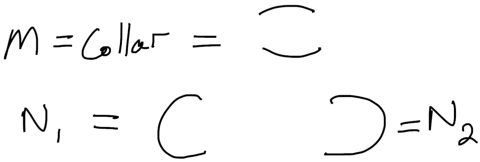

$$\int_{\bigcirc} A \simeq \int_{\subset} A \otimes_{\int_{=} A} \int_{\supset} A$$

$$\simeq \int_{-} A \otimes_{\int_{-} A \otimes \int_{-} A} \int_{-} A$$

$$\simeq A \otimes^{L}_{A \otimes^{L} A} A$$

Thank You! ^_^

- Alaya and Francis's *A Factorization Homology Primer*
- Alaya and Francis's original paper *Factorization Homology of Topological Surfaces*
- Cooke's thesis *Factorization Homology and Skein Categories of Surfaces*
- Lurie's lecture notes on *Nonabelian Poincaré Duality*
- Recordings of Tanaka's talks *Factorization Homology, ∞-Categories, and Topological Field Theories* (on youtube)