What Is Algebraic Geometry?

(And why should you care?)

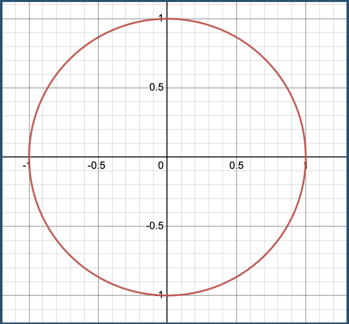
Chris Grossack
(they/them)

As usual, these slides will be available on my website

Grossack.site

§1

What is Algebraic Geometry?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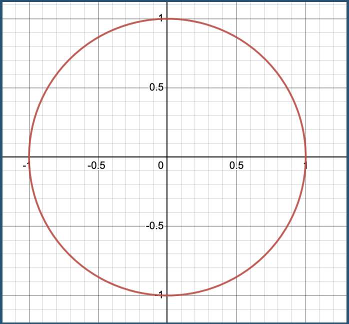
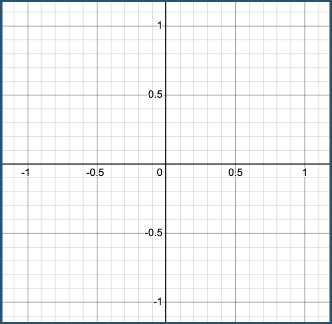


$$x^2 + y^2 = 1$$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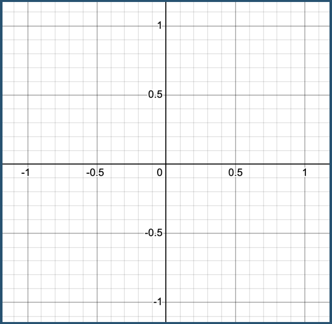
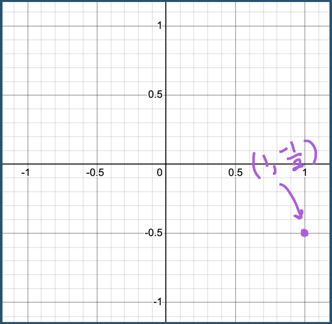


↳ each point gets a name



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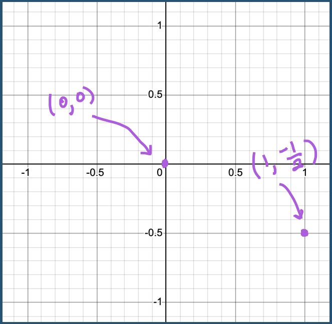

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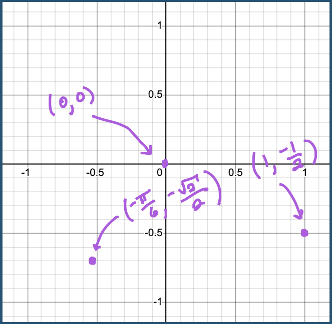

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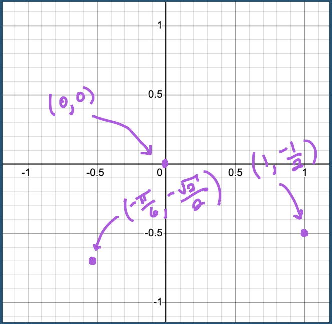

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↳ a general point is called (x, y)

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to denote a circle, we mean

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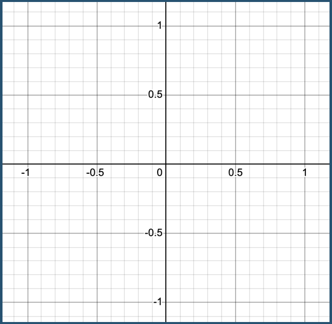

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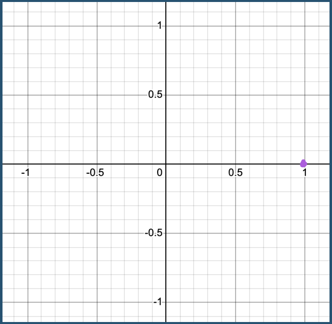

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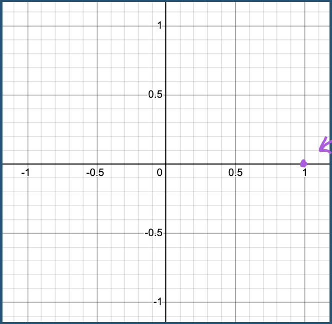

$(x,y)=(1,0)$

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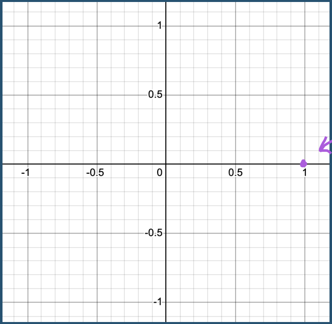

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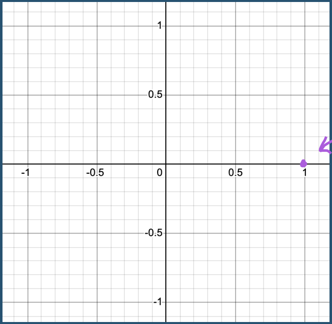

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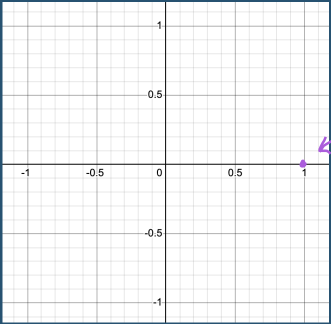

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(color this point)

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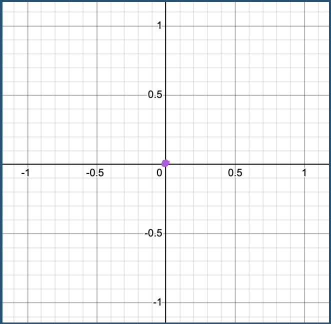

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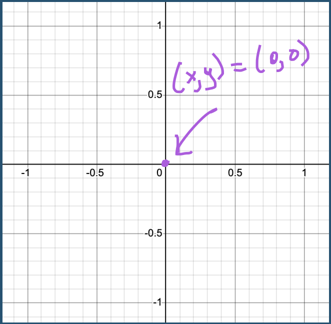

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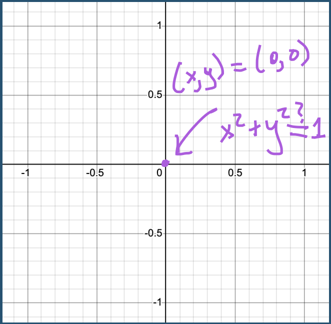

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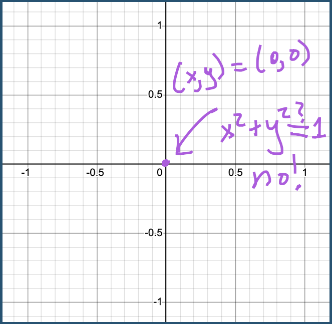

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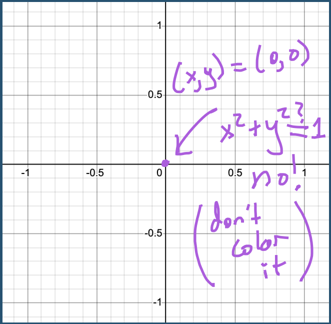

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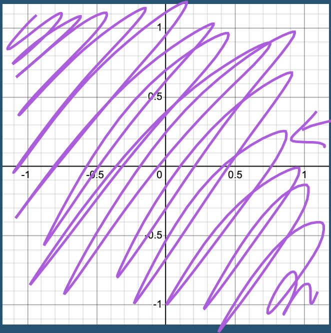

play this game for <u>every</u> point!

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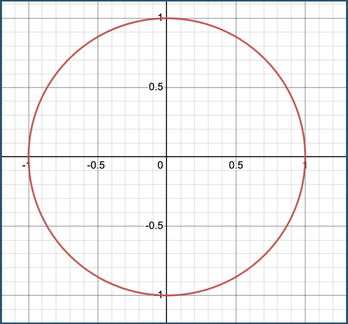

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- ↳ Every mathematician had to learn geometry in the ancient greek style
- ↳ This is really hard!

Eg :

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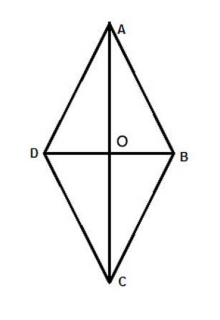

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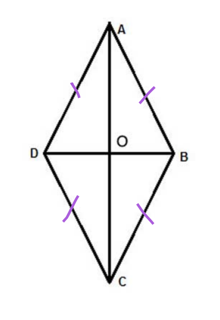

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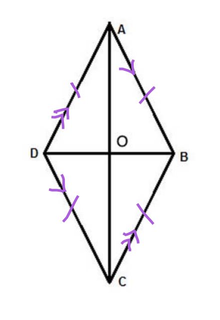

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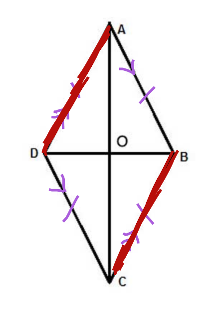

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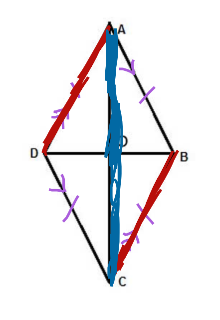

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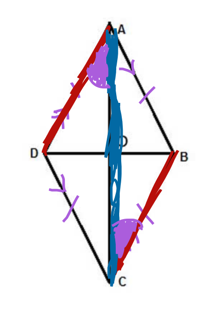

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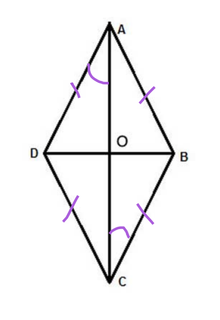

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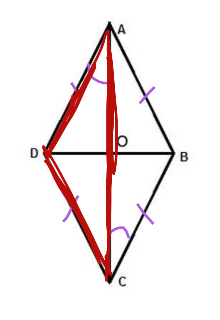

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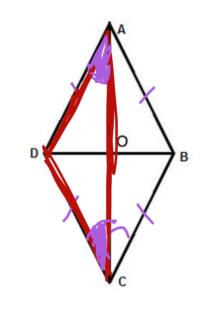

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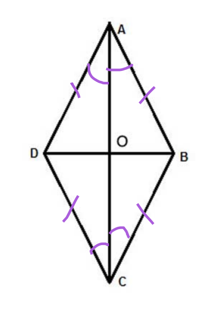

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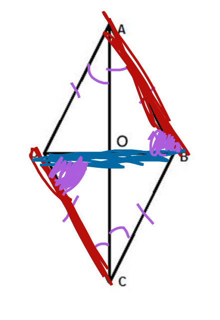

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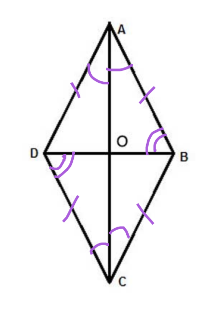

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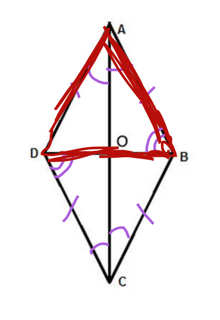

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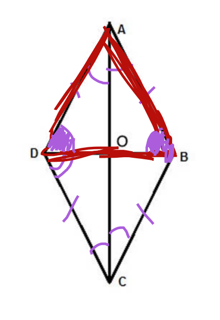

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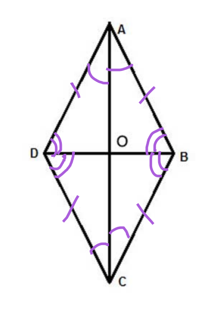

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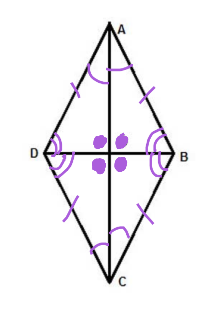

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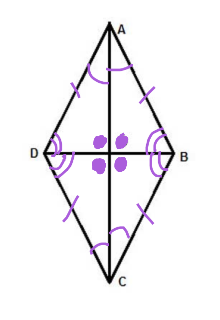

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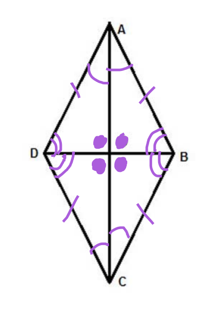

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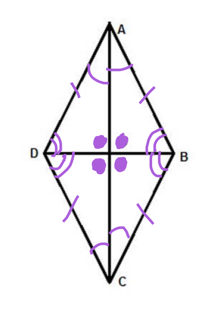

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So diagonals are $\perp$

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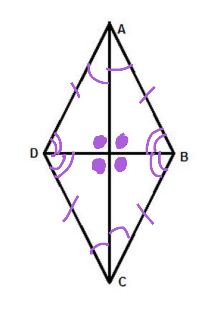

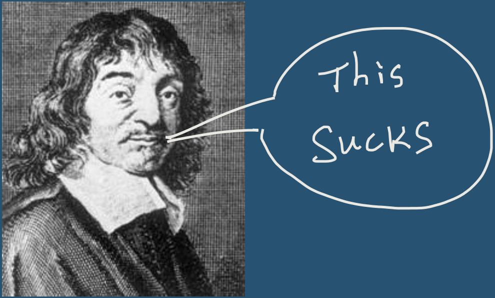

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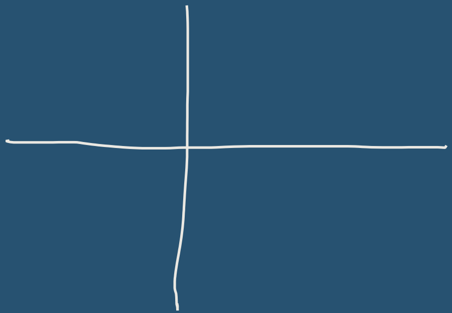

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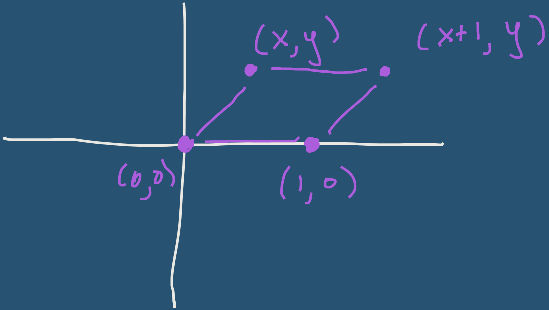

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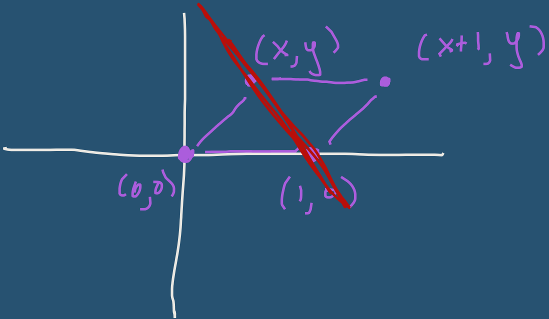

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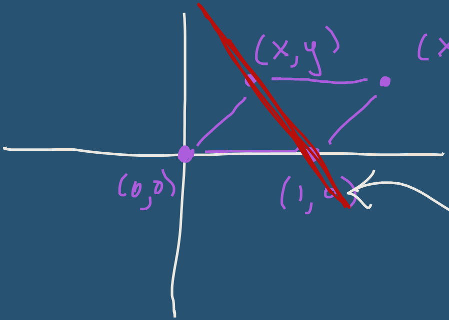

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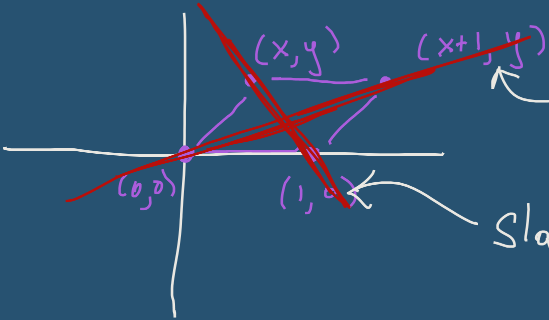

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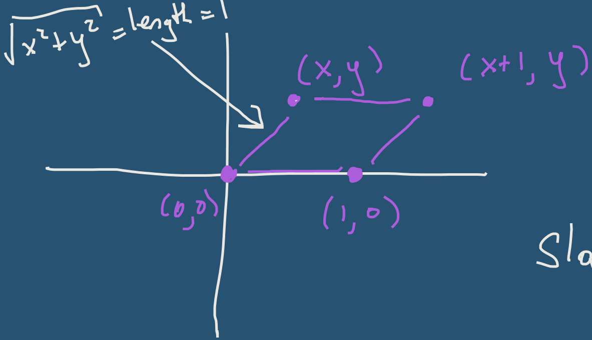

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Know $\underline{x^2+y^2=1}$ $\rightarrow x^2-1=-y^2$

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↳ So Descartes' idea let us <u>mechanize</u> geometry!

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- ↳ No need to be clever. Just add coordinates and solve some polynomial equations!

In fact, computers can solve <u>any</u> euclidean geometry problem automatically!

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↳ First proven by Tarski in 1951

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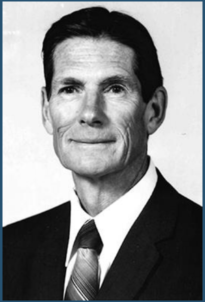
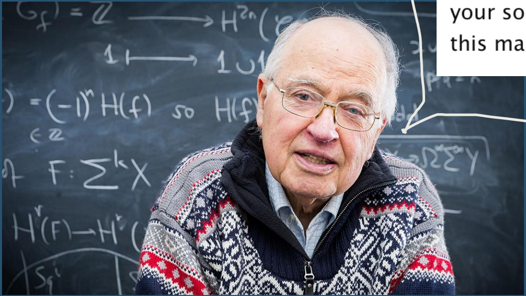
- ↳ First <u>efficient</u> procedure by Collins in 1975

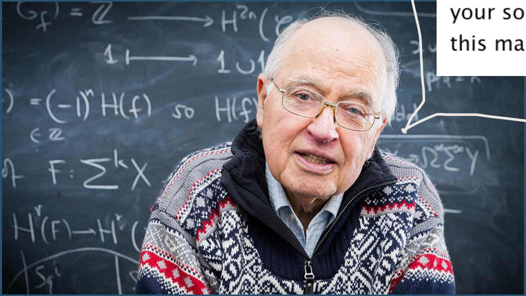
A. Tarski



G. Collins



Algebra is the offer made by the devil to the mathematician. The devil says: "I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine."



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– Sir Michael Atiyah

↳ what does Atiyah mean by this?

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↳ aren't we doing geometry?

↳ what does Atiyah mean by this?

↳ aren't we doing geometry?

↳ yes, but it's easy to forget!

Proposition 7.5. *Let X be a projective scheme over a field k. Then X has a dualizing sheaf.*

PROOF. Embed X as a closed subscheme of $P = \mathbf{P}_k^N$ for some N, let r be its codimension, and let $\omega_X^\circ = \mathscr{E}xt_P^r(\mathcal{O}_X,\omega_P)$. Then by (7.4) we have an isomorphism for any $\mathcal{O}_X$-module $\mathscr{F}$,

$$\operatorname{Hom}_X(\mathscr{F},\omega_X^\circ) \cong \operatorname{Ext}_P^r(\mathscr{F},\omega_P).$$

On the other hand, when $\mathscr{F}$ is coherent, the duality theorem for P (7.1) gives an isomorphism

$$\operatorname{Ext}_P^r(\mathscr{F},\omega_P) \cong H^{N-r}(P,\mathscr{F})'.$$

But $N - r = n$, the dimension of X, and $\mathscr{F}$ is a sheaf on X, so we obtain a functorial isomorphism, for $\mathscr{F} \in \mathfrak{Coh}(X)$,

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In particular, taking $\mathscr{F} = \omega_X^\circ$, the element $1 \in \operatorname{Hom}(\omega_X^\circ,\omega_X^\circ)$ gives us a homomorphism $t: H^n(X,\omega_X^\circ) \to k$, which we take as our trace map. Then it is clear by functoriality that (ω_X°,t) is a dualizing sheaf for X.

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ostensibly geometry!

More down-to-earth:

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let's solve
$$\begin{aligned} 2x + y - z &= 0 \\ x - y - z &= 0 \end{aligned}$$

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$\leadsto$ for <u>every</u> t,

$$x = \frac{2}{3}t \quad y = \frac{-1}{3}t \quad z = t$$

is a solution

what the hell did we just DO?

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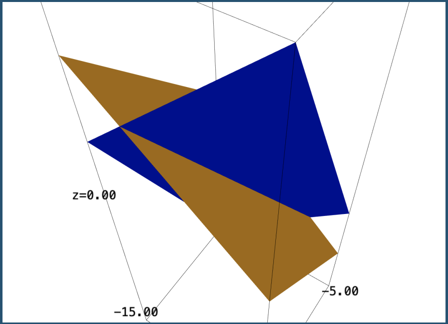

Blue: $2x+y-z=0$

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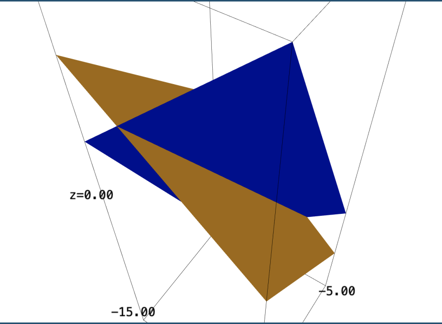

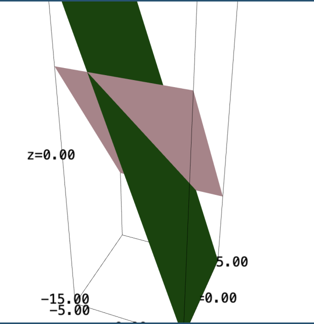

Blue: $2x+y-z=0$

Orange: $x-y-z=0$

pink: $z=\frac{3}{2}x$

green: $z=-3y$

↳ We found a <u>different</u> (simpler!) pair of planes whose intersection is the same as the intersection we started with!

↳ we found a <u>different</u> (simpler!) pair of planes whose intersection is the same as the intersection we started with!

↳ almost no linear algebra class will tell you this!

This is the secret to quickly "guessing" what's true without needing to calculate!

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Don't get lost in the sauce! Picture the geometry!

Eg:

Why is $\det(AB) = \det(A) \cdot \det(B)$?

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$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix}\right)$$

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(also it's negative for reflections)

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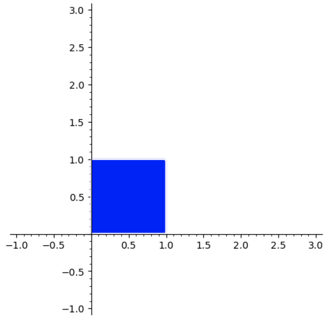

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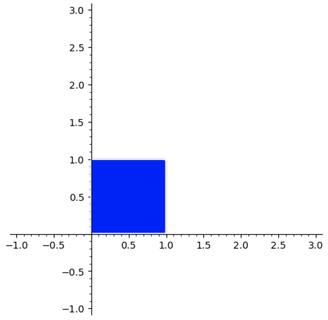
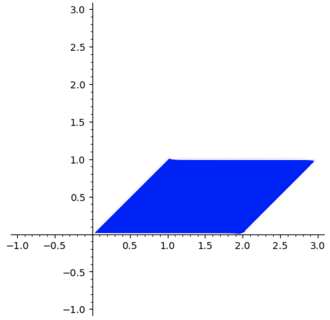


$\xrightarrow{A}$



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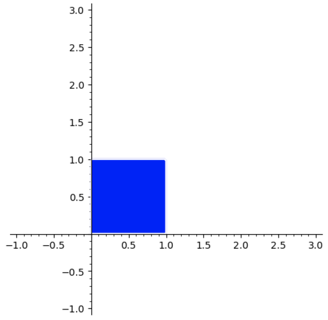

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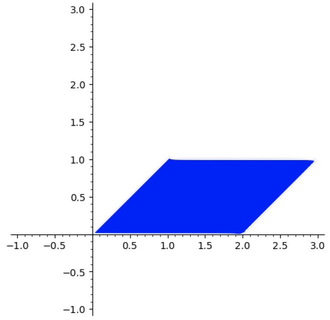

area$(\blacksquare) = 1$

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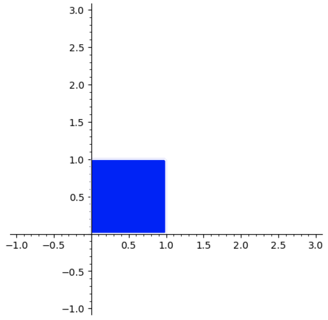

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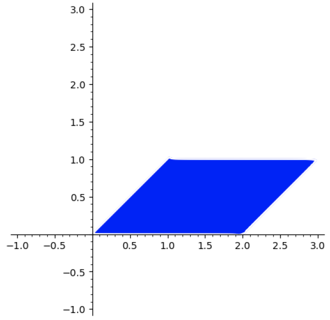

$\text{area}(\blacksquare) = 1 \qquad \text{area}(A\,\blacksquare) = 2$

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Applying A <u>doubled</u> the area.

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Applying A doubled the area.

So $\det(A) = 2$.

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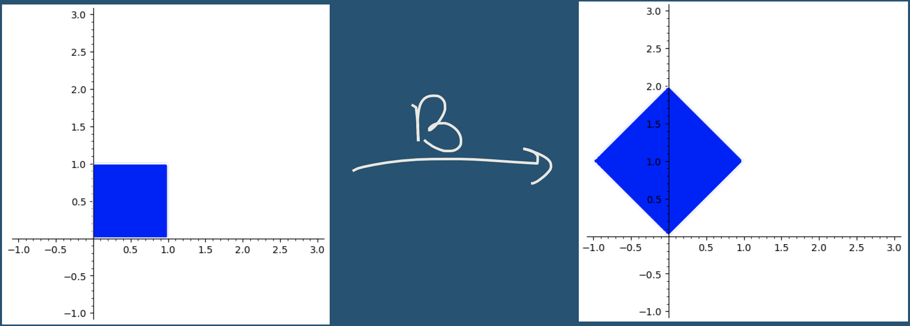

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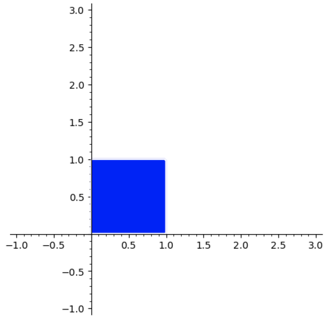
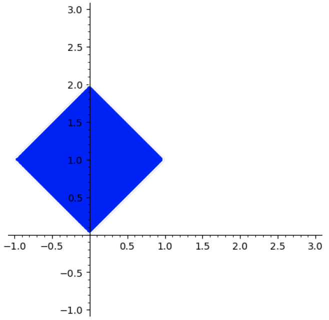


$\xrightarrow{B}$



area(▨) = 1

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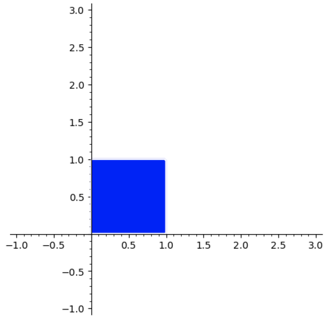
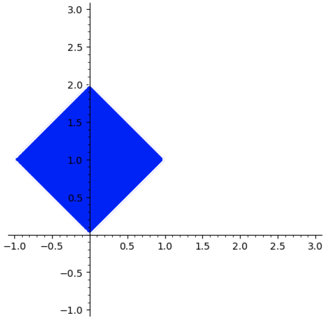
$\det(A) = 2.$



$\xrightarrow{B}$



$\text{area}(\square) = 1 \qquad \text{area}(B\square) = 2$

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So applying B doubles the area.

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So, geometrically, what <u>should</u> $\det(AB)$ be?

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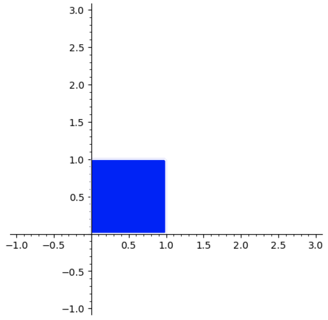
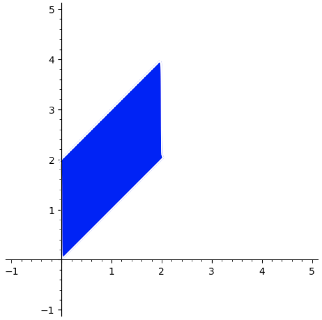


$\xrightarrow{AB}$



$\text{area}(\square) = 1$ $\qquad$ $\text{area}(AB\,\square) = 4$

↳ So this seemingly <u>archaic</u> algebraic fact is reflecting a <u>simple</u> geometric fact!

§2

Ok, but really...

what's algebraic geometry?

↳ Algebraic Geometry is this... but more!

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↳ First, more algebra was developed to describe more complicated geometric ideas

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↳ Rings, invariant theory, etc.

↳ Then, following Grothendieck, we developed more complicated geometric objects to help us understand the algebra!

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↳ schemes, stacks, topoi, etc.

Algebraic Geometry has a fearsome reputation, because the geometric objects seem quite far removed from any "actual geometry"

An Anachronistic History

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↳ $x^2+y^2=1$ is a circle

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↳ so is $x^2+y^2=4$, and $(x-3)^2+(y-1)^2=11$, etc.

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↳ these have the same geometry, and our algebra should reflect that.

An Anachronistic History

- $x^2+y^2=1$ is a circle
- so is $x^2+y^2=4$, and $(x-3)^2+(y-1)^2=11$, etc.
- these have the same geometry, and our algebra should reflect that.
- Solution? Rings!

An Anachronistic History

$\hookrightarrow \quad \dfrac{\mathbb{R}[x,y]}{x^2+y^2=1} \cong \dfrac{\mathbb{R}[x,y]}{x^2+y^2=4} \cong \cdots$

An Anachronistic History

↳ $$\frac{\mathbb{R}[x,y]}{x^2+y^2=1} \cong \frac{\mathbb{R}[x,y]}{x^2+y^2=4} \cong \cdots$$

↳ So ring theory develops to describe geometry.

An Anachronistic History

↳ But some rings don't come from geometry!

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↳ eg, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$

An Anachronistic History

↳ But some rings don't come from geometry!

↳ eg, $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$

↳ Can we <u>find</u> (more complex) geometric objects to describe these examples?

An Anachronistic History

An Anachronistic History

A. Grothendieck

An Anachronistic History

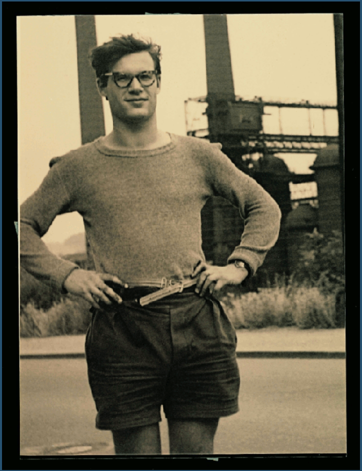
that's a real picture!

An Anachronistic History

that's a real picture!

So is this one:

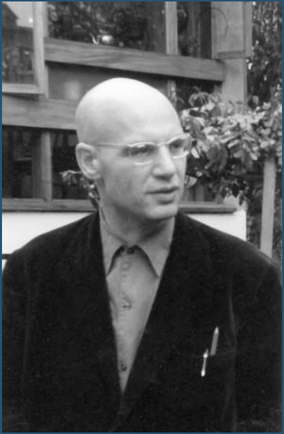
An Anachronistic History



An Anachronistic History

And this one....

An Anachronistic History



An Anachronistic History

↳ Grothendieck and his School show that these more general geometric objects still behave "geometrically"

An Anachronistic History

↳ Grothendieck and his School show that these more general geometric objects still behave "geometrically"

↳ This lets us do geometry in a much broader setting!

An Anachronistic History

↳ Caveat: These "schemes" are harder to visualize.

An Anachronistic History

- ↳ Caveat: These "schemes" are harder to visualize.
- ↳ But there are tricks, and it can be done.

An Anachronistic History

- Caveat: These "schemes" are harder to visualize.
- But there are tricks, and it can be done.

- eg. $\mathbb{Z}$ is a "smooth curve", with one point for every prime

An Anachronistic History

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↳ eg. $\mathbb{Z}$ is a smooth curve, with one point for every prime

↳ the curve itself "is" 0

An Anachronistic History

↳ But now we want to do geometry to these new objects!

An Anachronistic History

↳ But now we want to do geometry to these new objects!

↳ glue them together, intersect them, look for symmetries, etc

An Anachronistic History

↳ It turns out that, to study symmetries of these schemes, we need even more abstract objects (called "stacks")

An Anachronistic History

↳ So newer, harder, algebra is developed to study stacks, which are quite hard to visualize.

An Anachronistic History

↳ So newer, harder, algebra is developed to study stacks, which are <u>quite</u> hard to visualize.

↳ But it <u>can</u> be done!

An Anachronistic History

↳ This cat and mouse game is still underway!

An Anachronistic History

- This cat and mouse game is **still** underway!

- geometers want high power algebra to describe and solve their problems

An Anachronistic History

↳ This cat and mouse game is <u>still</u> underway!

↳ Algebraists want abstract notions of "geometry" so they can visualize their problems!

An Anachronistic History

↳ It's getting absurd.

An Anachronistic History

↳ It's getting absurd.

↳ see, eg, Lurie's

"Derived Algebraic Geometry"

An Anachronistic History

So we're caught up to today

An Anachronistic History

So we're caught up to today

with difficult and arcane algebra inextricably linked with abstract gadgets that, with lots of practice, you can visualize.

§ 3

Why the #@$* should anyone care?

Low Abstraction

Low Abstraction

↳ Solving polynomial equations

Low Abstraction

↳ Solving polynomial equations

↳ used constantly

Low Abstraction

↳ Solving polynomial equations

↳ used constantly

↳ "Gröbner Bases"

Low Abstraction

↳ Solving polynomial equations

  ↳ used constantly

  ↳ "Gröbner Bases"

↳ Quickly guessing true facts about polynomial systems.

Medium Abstraction

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Medium Abstraction

- Applying geometric intuition to purely algebraic settings
  - $\mathbb{Z}$ "is a curve"
  - Useful for the algebraists and number theorists.

High Abstraction

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↳ doing geometry to this lets us study how these spaces are related.

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High Abstraction $\neq$ Low Application

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↳ Hard problems need hard tools

↳ Hard problems can <u>also</u> be easy to understand!

eg. Fermat's Last theorem.

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↳ Are there integers x, y, z, n with $n \geq 3$ and

$$x^n + y^n = z^n \text{ ?}$$

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↳ Are there integers x, y, z, n with $n \geq 3$ and

$$x^n + y^n = z^n \, ?$$

↳ No! But the proof crucially uses "high abstraction" geometry.

Another eg:

↳ The Weil Conjectures

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↳ Predicts the number of solutions to a polynomial over $\mathbb{Z}/p^n\mathbb{Z}$

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↳ The Weil Conjectures

↳ Predicts the number of solutions to a polynomial over $\mathbb{Z}/p^n\mathbb{Z}$

↳ Required VERY high abstraction tools.

§4

Dipping Our Toes into Arithmetic Geometry.

Now is the time to
tune back in if
you want to ^_^

Why might the number theorists care about Algebraic Geometry?

Simple Question:

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↳ $3^2 + 4^2 = 5^2$

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↳ etc.

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↳ How could someone find this?

↳ Can we understand <u>all</u> pythagorean triples?

Answer:

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Yes! Through the power of geometry!

$$a^2 + b^2 = c^2$$

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$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

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$x, y \in \mathbb{Q} \longrightarrow x^2 + y^2 = 1$

So we want rational points

(x, y) so that $x^2 + y^2 = 1 \ldots$

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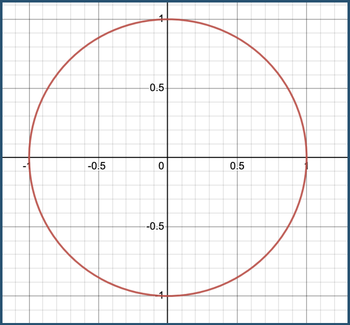

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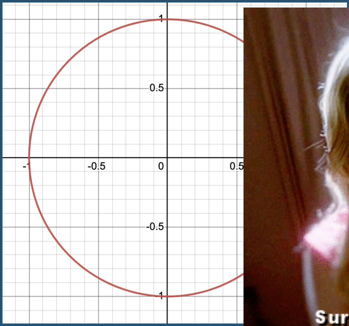

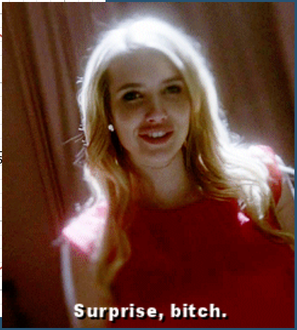

Now we take some known rational point (say, $(-1,0)$) and use it to get others!

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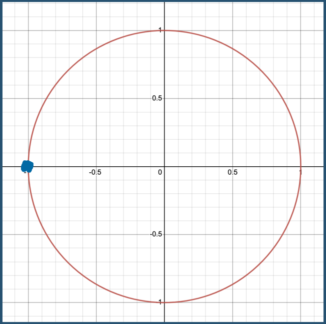

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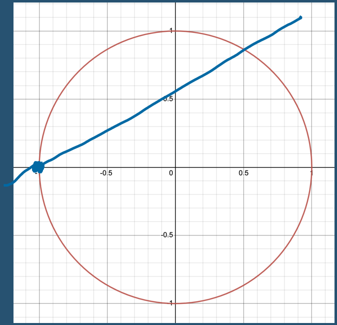

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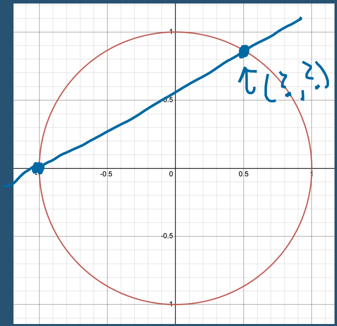

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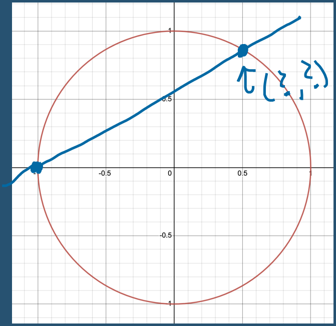

Slope $= t$

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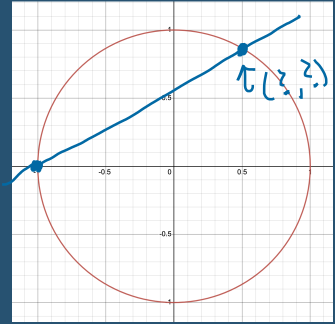


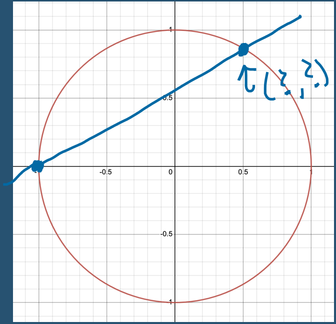
Slope $= t$

Solve

$$x^2+y^2=1$$

$$y-0=t(x+1)$$

Now we take some known rational point (say, (-1, 0)) and use it to get others!



Slope = t

$$(x,y) = \left\{ \begin{array}{l} (-1,0) \\ \quad\text{or} \\ \left(\dfrac{1-t^2}{1+t^2}, \dfrac{2t}{1+t^2}\right) \end{array} \right.$$

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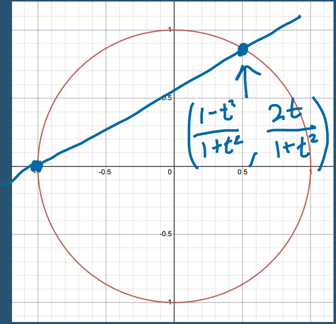

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$$\left(\frac{a}{c}, \frac{b}{c}\right) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

So we get a pythagorean triple

$$(1-t^2)^2 + (2t)^2 = (1+t^2)^2$$

Of course, any triple gives us a point on the circle

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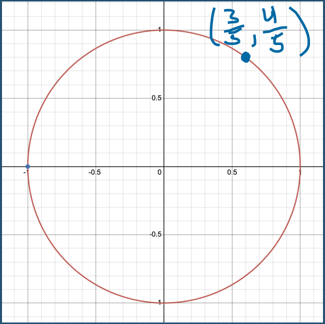

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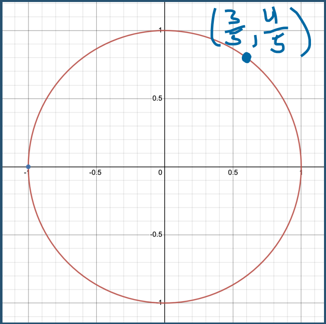

Thus a line to $(-1, 0)$

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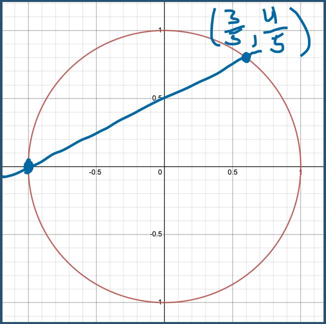

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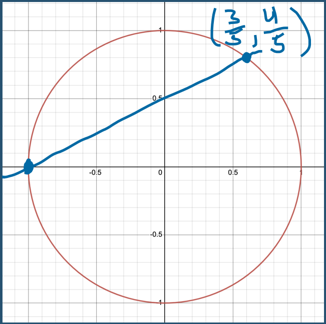

Thus a line to $(-1, 0)$

And this line has some slope t.

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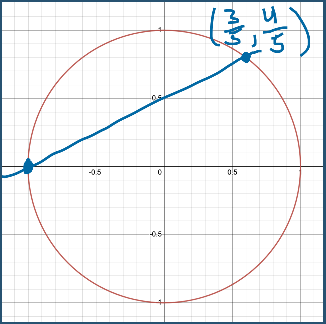

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$\hookrightarrow$ So this gets us <u>every</u> triple!

Thank You ^_^