Categories, Modalities, and Type Theories: Oh My

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(they/them)

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- Let's talk about the interplay between Categories and Modalities.
- There are lots of interesting connections here, but we're going to focus on two particularly simple case studies:
- How can we use Categories to talk about Modal Logic?
- How can we use Modal Logic to talk about Categories?
- I'll be assuming some basic category theoretic knowledge throughout, but I'll try to stay as approachable as possible.
- Officially I'm not assuming any knowledge of modal logic, but I'll be going over the basics as though it's a review.
- Let's get started!

Recall (Basic, Propositional) Modal Logic allows for all the (propositional) comforts you know and love

- Variables p
- “and” $\wedge$
- “or” $\vee$
- “not” $\neg$
- “implies” $\rightarrow$
- etc.

Plus a few $\sim\star\sim$ Bonus Connectives $\sim\star\sim$ called Modalities.

- “box” $\Box$
- “diamond” $\Diamond$

The "basic" part of the name is becasue we're only adding one pair of modalities. In general we could add a whole family $\Box_i$ and $\Diamond_i$.

The interpretation of these symbols varies, but it is traditional to introduce them as

- $\Box\varphi \rightsquigarrow$ "φ is necssary"
- $\Diamond\varphi \rightsquigarrow$ "φ is possible"

Other interpretations include

$\Box\varphi$	$\Diamond\varphi$
Valerie knows φ	Valerie thinks φ is possible
Henceforth φ	At some point φ
φ is provable	φ is not disprovable

- Notice the truth of $\Box\varphi$ depends on φ!
- Contrast this with $\neg$, which only depends on the *truth value* of φ
- Modalities are useful because they model connectives that aren't "truth functional"

How do $\Box$ and $\Diamond$ behave?

- Duality: $\Diamond\varphi \equiv \neg\Box\neg\varphi$
- K: $\Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi$
- Necessitation: $\dfrac{\varphi}{\Box\varphi}$

We can also add bonus axioms to indicate certain features our particular modality should posess. Important for our purposes include:

- T: $\Box\varphi \to \varphi$ (equivalently, $\varphi \to \Diamond\varphi$)
- 4: $\Box\varphi \to \Box\Box\varphi$ (equivalently, $\Diamond\Diamond\varphi \to \Diamond\varphi$)
- S4: An abbreviation for T $+$ 4

And what about semantics?

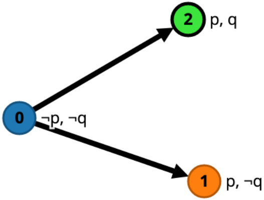
Definition

A Frame $\mathfrak{F}$ is a set W equipped with a binary relation $R \subseteq W \times W$.

A Model $\mathfrak{M}$ is a frame $\mathfrak{F}$ equipped with a Valuation Function $[\![\cdot]\!] : \mathrm{Prop} \to 2^W$

You should think of W as a set of "possible worlds", and $w \in [\![p]\!]$ as saying p is true in the world w.

Our next step is to *extend* $[\![\cdot]\!]$ to all formulas. First, some intuition:



The arrows tell us how the possible worlds are related.

You might imagine φ is *necessary* if in all the worlds I can see, φ is true. So we might want $0 \models \Box p$

⚠Since 0 doesn't see itself, it doesn't matter that $0 \models \neg p$.

Similarly, if φ is true in *some* world I can see, we should think φ is *possible*.

So $0 \models \Diamond q \wedge \Diamond \neg q$

Notice $1 \models \Box \bot$ (vacuously), since it doesn't see *any* worlds at all.

Formally...

Definition

We recursively extend $[\![\cdot]\!]$ to all formulas as follows:

- $[\![p]\!]$ is defined as part of the model
- $[\![\varphi \wedge \psi]\!] \triangleq [\![\varphi]\!] \cap [\![\psi]\!]$
- $[\![\neg\varphi]\!] \triangleq [\![\varphi]\!]^c$
- $[\![\Box\varphi]\!] \triangleq \{w \mid \forall wRw'.w' \in [\![\varphi]\!]\}$
- All other connectives are defined in terms of these using duality. For concreteness, though:
- $[\![\Diamond\varphi]\!] \triangleq \{w \mid \exists wRw'.w' \in [\![\varphi]\!]\}$

Definition

We then define $w \models \varphi$
(or $\mathfrak{M}, w \models \varphi$, if the model is not clear from context)
as $w \in [\![\varphi]\!]$.

- (Propositional) Modal logic perfectly balances utility against simplicity
- It is expressive enough to model a lot of real world scenarios
- Yet it is robustly decidable – almost any reasonable changes you make will still give you a decidable logic, which admits model checking, and lots of other nice features.
- This makes it extremely useful in applications, and even in industry
- I can't help but wonder, though. . . What about function symbols? Doesn't it seem restrictive to be working in the propositional world all the time?
- And now, a seemingly unrelated topic.

Let $\mathcal{C}$ be a (small) category.

Recall the Presheaf Category $\mathsf{Set}^{\mathcal{C}}$ has lots of structure, no matter how little structure $\mathcal{C}$ started with!

In fact, following Makkai and Reyes, it has enough structure to interpret First Order Logic (and indeed, higher order logics too) inside it.

Let's see a simple example:

Say $\mathcal{C} = A \xrightarrow{f} B$ is the "walking arrow". Then a group in $\mathrm{Set}^{\mathcal{C}}$ is a pair of groups G_A and G_B, plus a distinguished homomorphism between them:

$$G_A \xrightarrow{G_f} G_B$$

In general, a model of some theory T in $\mathrm{Set}^{\mathcal{C}}$ looks like a picture of $\mathcal{C}$ made entirely of T-models. That is, a functor from $\mathcal{C} \to \mathrm{Mod}(T)$.

You can chase through the definitions to do this, or you can use Lawvere's idea of "Functorial Semantics" to make it almost immediate.

We should think of a model in $\mathrm{Set}^{\mathcal{C}}$ as being a regular model that is "changing with time", or which "varies along $\mathcal{C}$"... Does this feel modal to anyone else?

⚠A quick caveat lector: This is too natural to have not been studied before, but the idea is my own. I haven't had time to fully establish a proof system, let alone to check any soundness or completeness results. Because of that, it's technically possible that everything I'm about to say is a lie... I would be very surprised, though. If anyone knows someone else who has studied results like these before, I would love to hear about it.

Say $\mathcal{C}$ is a frame, viewed as a category.

Then $\mathrm{Set}^{\mathcal{C}}$ provides a natural semantics for First Order Modal Logic!

We have a *family* of "classical" models, parametrized by our frame.

We know how to interpret the first order fragment of our logic at any particular world

and the modal operators $\Box$ and $\Diamond$ give us a way of seeing what nearby models think.

Let's see an example

Consider the group

$$\langle x \rangle \xleftarrow[\quad]{\substack{x^2 \leftarrow\!\shortmid\, a \\ x^2 \leftarrow\!\shortmid\, b}} \langle a, b \rangle \xrightarrow[\quad]{\substack{a \mapsto y \\ b \mapsto y}} \langle y \rangle$$

in $\mathrm{Set}^{\bullet \leftarrow \bullet \rightarrow \bullet}$.

Then

- $\langle a, b \rangle \models \Box.\forall x.\forall y.xy = yx$
- $\langle a, b \rangle \models \forall x.\Diamond.\exists y.y^2 = x$

As an aside, I'll tell you something that has *definitely* been studied before. This way I'm guaranteed to say at least one true thing in this talk.

Instead of working with presheaf models, we can work with étale models instead (see Awodey and Kishida, 2008).

The idea is to have a T-model in each fibre, then for each p in the base space we can say p models some first order formula exactly when the model sitting over it does.

As for the modal operators, we say that $p \models \Box\varphi$ exactly when p has a neighborhood with every point in that neighborhood satisfying φ.

There is almost certainly a way to make sense of the previous slide as a special case of this machinery, but I haven't had time to think it through.

Intermission

We've just seen how categories can help us study modal logic

Can we use modal logic to help us study categories?

The answer is a resounding "yes!", and this is an extremely active area of research.

Before we can get into it, we need to take a second to talk about Type Theories

To each category $\mathcal{C}$, we can associate a "programming language" $T_{\mathcal{C}}$

- It has a type for each object of $\mathcal{C}$
- It has a term of type B (with a variable in A) for each arrow $f : A \to B$ in $\mathcal{C}$

There's a close correspondence between features of our programming language and structure on our category.

Category	Type Constructor
Products	$\times$-types
Cartesian Closed	$\to$-types
Regular	subtypes $+ \exists + \wedge$
Topos	full FOL, "powerset"-types, etc.

Oftentimes we find ourselves with a distinguished (co)monad on $\mathcal{C}$ that we want to study.

What kind of constructor do we add to our programming language in order to have access to this (co)monad?

Well, what even is a monad? Does that tell us anything?

Definition

A Monad on $\mathcal{C}$ is a functor $M : \mathcal{C} \to \mathcal{C}$ equipped with:

- $\eta : 1_{\mathcal{C}} \Rightarrow M$
- $\mu : M^2 \Rightarrow M$

satisfying certain natural coherence conditions.

Definition

A Comonad on $\mathcal{C}$ is a functor $W : \mathcal{C} \to \mathcal{C}$ equipped with:

- $\epsilon : W \Rightarrow 1_{\mathcal{C}}$
- $\nu : W \Rightarrow W^2$

satisfying certian natural coherence conditions.

Let's expand out the definitions, and see if we notice anything.

For each type A, a monad gives us

- $\eta_A : A \to MA$
- $\mu_A : MMA \to MA$

If we suggestively write $\Diamond$ for M, then we see exactly the axioms for S4!

Dually, a comonad gives us

- $\epsilon_A : WA \to A$
- $\nu_A : WA \to WWA$

Again, we recognize these as the S4 axioms where W plays the role of $\Box$.

"What about K!?", I hear you asking... It's a bit complicated.

To even *talk* about an arrow $M(A \to B) \to MA \to MB$, we need cartesian closed structure on $\mathcal{C}$. That way we can look at $M(B^A)$.

Unfortunately, even though morally M satisfies K by functoriality, there's no reason this should "internalize" to a map $M(B^A) \to MB^{MA}$. There's a couple ways around this.

- Following Moggi 1991, we could restrict attention to "strong monads", which satisfy $M(A \times B) \cong MA \times MB$
- Following Kobayashi 1997, we could look at the more general class of "$\mathcal{L}$-strong monads", which satisfy a slightly more technical condition.
- Realizing we're running low on time, we could simply ignore this problem, safe in the knowledge that it can be solved.

I'm kidding, I wouldn't do that to you!

Kobayashi describes the system CS4 (for Constructuve S4) which has a proof system as follows:

- All theorems of intuitionistic propositional logic
- Modus Ponens
- K – $\Box(A \to B) \to \Box A \to \Box B$ $\Box(A \to B) \to \Diamond A \to \Diamond B$
- T – $\Box A \to A$ $A \to \Diamond A$
- 4 – $\Box A \to \Box\Box A$ $\Diamond\Diamond A \to \Diamond A$
- $\bot_E$ – $\Diamond\bot \to A$
- Necessitation for $\Box$

As usual we define $\neg A \equiv A \to \bot$, and duality between $\wedge$ and $\vee$ fails without DNE.

Perhaps unsurprisingly, then duality between $\Box$ and $\Diamond$ fails too. This is why we explicitly include the dualized axioms for K, T, and 4

- Kobayashi goes on to construct the syntax for a type theory corresponding to CS4, and prove that this type theory is sound and complete with respect to the "$\mathcal{L}$-strong monad" semantics from earlier.
- This is far from the last word on the intersection of modal logic and type theory
- People are using modalities to add features from differential geometry into HoTT
- The direction I find most exciting is the use of modalities to interface between HITs like S^1 and "topological" definitions of the same (like $\{x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$). See "Cohesive HoTT".

Thank you!