2-categorical Descent and (Essentially) Algebraic Theories

Chris Grossack
(they/them)

UC Riverside

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For example, the domain of $\circ$ is

$$\left\{(f,g) \in A \times A \mid \mathrm{dom}(f) = \mathrm{cod}(g)\right\}$$

which is defined by some equations in the total operations.

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Which is why it's good that...

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Unfortunately, they're not *quite* as nice as algebraic theories. . .

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This does *not* need to be true for models of essentially algebraic theories!

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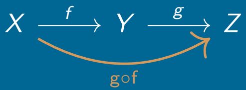

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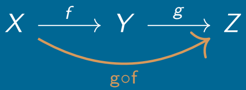

After quotienting the objects, our partial operation $\circ$ sees that f and g are composable! So we *must* add an arrow $g \circ f$

But this means, in the quotient, our set of arrows is $\{f, g, g \circ f\}$, which is *not* a quotient of our original set of arrows $\{f, g\}$!

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But *essentially* algebraic theories only admit models in a category with finite *limits*. (that is, we need equalizers too!) This is annoying if we want to interpret "smooth" versions of our algebras, since Diff famously lacks finite limits!

For instance, this is why a lie groupoid is *not* simply a groupoid object in Diff. Groupoids, special categories, are merely *essentially* algebraic!

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Both proofs are basically the same, and crucially use quite a lot of category theory!

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- The difference is *equalizers*

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Here we're thinking of the free construction $\mathrm{Eq}(-)$ as the left adjoint to the forgetful functor U from finite limit categories to finite product categories.

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That is, we want to understand the essential image of the Eq functor!

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The keyword is **Comonadicity** of the adjunction.

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Definition

We say $L \dashv R$ is **Comonadic** if $L : \mathcal{A} \to \mathcal{X}_{LR}$ is an equivalence. That is, if we can recognize objects of the form LA as precisely those objects X admitting an LR-coalgebra structure!

So if we can show that $\mathrm{Eq} \dashv U$ is comonadic, we'll be done!

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$L \dashv R$ is comonadic if and only if

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These conditions sound scarier than they are, and with the explicit definition of $\mathrm{Eq}(-)$ in a paper of Bunge-Carboni[1] it's not so hard to just explicitly check these conditions.

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Pedicchio and Wood push this further, and give a concrete description of the categories we're interested in! The key definition is that of "enough effective projectives".

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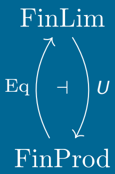

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$$\begin{array}{ccc} \mathrm{FinProd} & & \approx \text{algebraic} \\ \mathrm{Eq} \uparrow \quad \dashv \quad \downarrow U & & \\ \mathrm{SymMon} & & \approx \text{props} \end{array}$$

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But, using an explicit construction that Todd Trimble posted on the nlab forums, it should be possible to play the same game. But there's still lots of details to check.

If you want to read more, you'll likely be interested in

- Adámek, Vitale, and Rosický's *Algebraic Theories*
- Borceux's *Handbook of Categorical Algebra (Vol 2)*
- Bunge and Carboni's *The Symmetric Topos*
- Palmgren and Vicker's *Partial Horn Logic and Cartesian Categories*

and of course

- Pedicchio and Wood's *A Simple Characterization of Theories of Varieties*

Thank You!