Model Theory and You:

Infinite Theorems for Free

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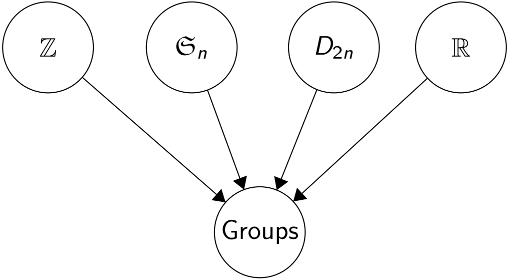
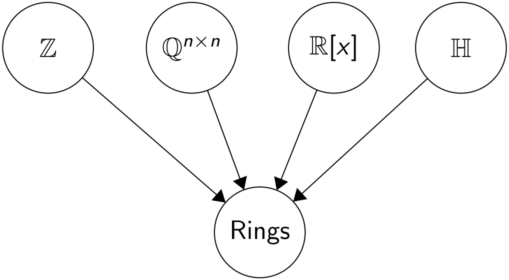
What is Model Theory?

What are groups?



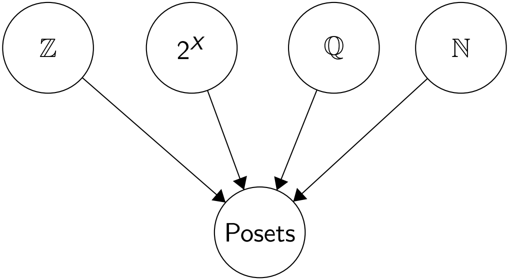
- We abstract the common features of many notions of "symmetry"

What are rings?



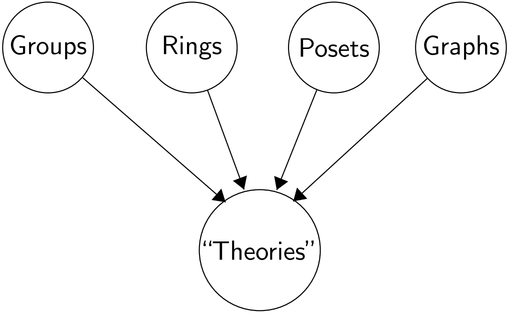
- We abstract the common features of many notions of “arithmetic”

What are posets?



- We abstract the common features of many notions of "order"

What is Model Theory?



- We abstract the common features of many notions of "structure"

Objects in math often come pre-equipped with certain symbols that we use to study them. For example:

- $\sigma_{\text{Group}} = (e, {}^{-1}, \cdot)$
- $\sigma_{\text{Ring}} = (0, 1, -, +, \times)$
- $\sigma_{\text{Ring (Bad)}} = (0, -, +, \times)$
- $\sigma_{\text{Poset}} = (\leq)$
- $\sigma_{\text{Arithmetic}} = (0, 1, +, \times, \leq)$
- $\sigma_{\text{Reals}} = (\mathbb{Q}, \pi, e, \text{etc.}, \{f : \mathbb{R}^n \to \mathbb{R} \text{ smooth}\}, \leq)$

Now we abstract.

Definition

A *signature* $\sigma = (C, F_\bullet, R_\bullet)$ is a triple with

- A set of *Constant Symbols* C
- A set of *Function Symbols* F_n for each $n \in \mathbb{N}$
- A set of *Relation Symbols* R_n for each $n \in \mathbb{N}$

But what do we do with it?

Definition (informal)

Given a signature σ, the *Language* $\mathcal{L}(\sigma)$ (sometimes written $\mathcal{L}_{\omega,\omega}(\sigma)$) is the smallest set of formulas built using the following symbols:

- Variables $x_1, x_2, \ldots$ (though we allow ourselves to write y, etc. too)
- Symbols from σ
- $=$
- $\wedge$
- $\vee$
- $\neg$
- $\rightarrow$
- $\leftrightarrow$
- $\forall$
- $\exists$

This definition *can* be made formal, but here is a definition by examples

- unitLaw $= (\forall x. xx^{-1} = e \wedge x^{-1}x = e) \in \mathcal{L}(\sigma_{\text{Group}})$
- isTransitive $= (\forall x.\forall y.\forall z. x \le y \wedge y \le z \to x \le z) \in \mathcal{L}(\sigma_{\text{Poset}})$
- isAbelian $= (\forall x.\forall y. xy = yx) \in \mathcal{L}(\sigma_{\text{Group}})$
- walk$_4(x, y) = (\exists z_1.\exists z_2.\exists z_3. xEz_1 \wedge z_1Ez_2 \wedge z_2Ez_3 \wedge z_3Ey) \in \mathcal{L}(\sigma_{\text{Graph}})$
- card$_2 = (\exists x_1.\exists x_2. x_1 \ne x_2 \wedge \forall y. y = x_1 \vee y = x_2) \in \mathcal{L}(\emptyset)$

Of course we don't yet know how to say if these formulas are "true" or "false". Let's remedy this.

Groups, Rings, Posets, etc. are all *sets* equipped with some extra *structure* (in the form of distinguished constants, functions, and relations).

We abstract.

Definition

Given a signature $\sigma = (C, F_\bullet, R_\bullet)$, a σ-structure $\mathfrak{M}$ is a set M equipped with

- An element $c^{\mathfrak{M}} \in M$ for each $c \in C$
- A function $f^{\mathfrak{M}} : M^n \to M$ for each $f \in F_n$
- A subset $r^{\mathfrak{M}} \subseteq M^n$ for each $r \in R_n$

- $(\mathbb{Z}, 0, -, +)$ is a σ_{Group} structure
- $(\mathbb{Z}, 3, -, \times)$ is *also* a σ_{Group} structure
- $(2^X, \subseteq)$ is a σ_{Poset} structure
- $(\{1,2,3\}, \{(1,2),(2,3),(3,1)\})$ is *also* a σ_{Poset} structure.

Of course, for Groups, Posets, etc. we don't *just* have symbols at our disposal. There are *axioms* which we demand of those symbols.

That is, we want to ensure certain formulas are *true*.

We abstract.

What is Truth?

Definition (informal)

Given a σ-model $\mathfrak{M}$ and a formula $\varphi \in \mathcal{L}(\sigma)$, we say

$$\mathfrak{M} \models \varphi$$

(read "$\mathfrak{M}$ *models* φ", "$\mathfrak{M}$ *satisfies* φ", "$\mathfrak{M}$ *thinks that* φ *is true*", etc.) if φ is actually true when we interpret it in "the obvious way" using $\mathfrak{M}$.

Similarly, if $\varphi(x_1, \ldots, x_n)$ has free variables, and $a_1, \ldots, a_n \in M$, we say

$$\mathfrak{M} \models \varphi(a_1, \ldots, a_n)$$

if φ is true when we substitute each a_i for x_i.

Again, this *can* be made formal. But we'll give a definition by examples.

- $(\mathbb{Z}, 0, -, +) \models \forall x.xe = x \wedge ex = x$
- $(\mathbb{Z}, 3, -, \times) \not\models \forall x.xe = x \wedge ex = x$
- $(\mathbb{N}, \leq) \models \exists x.\forall y.x \leq y$
- If $\varphi(x) = \exists y.x = y + y$, then
  - $(\mathbb{Z}, 0, -, +) \models \varphi(2)$
  - $(\mathbb{Z}, 0, -, +) \models \neg\varphi(1)$

Now that we have *truth*, we can talk about axioms for a theory.

Definition

If $A \subseteq \mathcal{L}(\sigma)$, then we say

$$\mathfrak{M} \models A$$

if and only if $\mathfrak{M} \models \varphi$ for every $\varphi \in A$. In this case, we say $\mathfrak{M}$ is an A-model.

Moreover, we say A "has a model" if and only if some $\mathfrak{M} \models A$.

$$A_{\text{Group}} = \left\{ \begin{array}{c} \forall x. ex = x \wedge xe = x \\ \forall x. xx^{-1} = e \wedge x^{-1}x = e \\ \forall x. \forall y. \forall z. (xy)z = x(yz) \end{array} \right\}$$

Fundamental Question

When does a set of axioms A have a model?

Obvious Condition

φ and $\neg\varphi$ cannot *both* be in A.
i.e. if there's two constant symbols c and d, no model can satisfy both $c = d$ and $c \neq d$.

Slightly Less Obvious Condition

We shouldn't be able to *derive a contradiction* from A.
i.e. we can't have both $\exists y.\forall x.x = y$ and $\exists x.\exists y.x \neq y$, even though they aren't *directly* negations of each other.

Definition Sketch

A *derivation* from A is a finite list of formulas

$$\varphi_1, \varphi_2, \ldots, \varphi_n$$

such that each φ_i satisfies one of the following

- an axiom from A
- one of the (finitely many) "structural axioms" such as $\varphi \vee \neg\varphi$
- $\psi \to \varphi_i$ and ψ both show up earlier in the list.

Important remark: Even if $|A|$ is infinite, any derivation from A can use only finitely many axioms from A.

Theorem (Gödel's Completeness Theorem)

The obvious obstruction is the only one.
A has a model if and only if there is no derivation of $\varphi \wedge \neg\varphi$ from A.

Fundamental Corollary (Logical Compactness)

A has a model if and only if every finite $A_0 \subseteq_{\text{fin}} A$ has a model

Proof.

If A has a model, then that same model works for each $A_0 \subseteq A$.

If A doesn't have a model, then by completeness, we can derive a contradiction from A. But that derivation can refer to only finitely many axioms. Then this finite subset of A also derives a contradiction, and has no model. □

This is probably the most important tool in model theory.

Let's see some applications.

Theorem

If A has arbitrarily large finite models, then A has an infinite model.

Proof.

We add countably many new constant symbols c_i to the signature of A. We look at the theory $A^* = A \cup \{c_i \neq c_j \mid i \neq j\}$. Notice if A^* has a model, then it will be a model of A which has infinitely many elements. Let A_0 be a finite subset of A^*. By compactness it suffices to show that A_0 has a model.
But A_0 is finite, and thus can only refer to finitely many of the $c_i \neq c_j$ axioms. Pick a model of A with enough elements to model these axioms, and assign the other constants to any element at all.
This is an A_0 model, proving the claim. □

Theorem

Let Γ be a graph. If every finite subgraph is k-colorable, then so is Γ.

Proof.

Let σ be a signature with

- A constant c_v for each vertex v in Γ
- A binary relation E
- k new constants $r_1, \ldots, r_k$
- A unary function symbol f

We the look at the axioms

- $c_v \neq c_w$ for each $v \neq w$ in Γ
- $c_v E c_w$ if and only if v and w are adjacent in Γ
- $\neg c_v E c_w$ if and only if v and w aren't adjacent in Γ
- $\forall x. f(x) = r_1 \vee f(x) = r_2 \vee \ldots \vee f(x) = r_k$
- $\forall x. \forall y. xEy \to f(x) \neq f(y)$

We do the same trick. Any finite subset A_0 of these axioms refers to finitely many c_v. The (finite!) induced subgraph on these v will model A_0 by choosing f to be a k-coloring. The claim follows by compactness. □

Using this same idea, we can prove the following theorems with a bit more bookkeeping:

- Every torsion-free abelian group admits an partial ordering $\leq$ which is compatable with the group structure. That is, whenever $a \leq b$, $ac \leq bc$ too.

  Hint: Look at the lexicographic order on $\mathbb{Z}^n$.

- If every finitely generated subgroup of G admits a faithful representation on $\mathbb{R}^n$, then so does G.

  Hint: Beware: This one is tricky. You'll want access to the lanuage of groups *and* the language of fields. You can describe $n \times n$ matrix multiplication using n^2 formulas, and you can also write down that a group homomorphism is injective.

And now for a completely different topic.

Grp, **Pos**, and so forth all form nice categories. Can we abstract this property too? What should the arrows between our structures look like?

Definition

A homomorphism of σ-structures $h : \mathfrak{M} \to \mathfrak{N}$ is a function $h : M \to N$ with the following bonus properties:

- $h(c^{\mathfrak{M}}) = c^{\mathfrak{N}}$ for each $c \in C$
- $h(f^{\mathfrak{M}}(x_1, \ldots, x_n)) = f^{\mathfrak{N}}(h(x_1), \ldots, h(x_n))$ for each $f \in F_n$
- $r^{\mathfrak{M}}(a_1, \ldots, a_n) \implies r^{\mathfrak{N}}(h(a_1), \ldots, h(a_n))$ for each $r \in R_n$

It is clear that the set of σ-structures and homs forms a category. Moreoever, for any set of axioms A, the set of A-models is a full subcategory.

An aside:

Have you ever wondered why we can check that f is a group homomorphism by only checking that $f(xy) = f(x)f(y)$? This is (a priori) only a *semigroup* homomorphism, but for some reason it preserves the whole group structure.

This is a special case of a model-theoretic phenomenon! Let's investigate.

Definition

A formula φ is called *positive* if it has no instances of $\neg$, $\to$, or $\leftrightarrow$.
This is beacuse $\varphi \to \psi$ is an abbreviation for $\neg\varphi \vee \psi$, and so it has a hidden $\neg$.

Theorem

Homomorphisms preserve all positive formulas.
That is, for every positive φ and every homomorphism $h : \mathfrak{M} \to \mathfrak{N}$, if $\mathfrak{M} \models \varphi(a_1, \ldots, a_n)$, then $\mathfrak{N} \models \varphi(h(a_1), \ldots, h(a_n))$.
In particular, if φ has no free variables, then

$$\mathfrak{M} \models \varphi \implies \mathfrak{N} \models \varphi$$

Proof.

An easy induction on the definition of formulas I omitted. □

- If f is a group hom and G is abelian, then $f[G]$ must be abelian. This is because $\forall x.\forall y.xy = yx$ is positive.
- The converse is false, though! If g and h don't commute in G, then $G \models gh \neq hg$. But the abelianization $p : G \to G^{ab}$ has $G^{\text{ab}} \models p(g)p(h) = p(h)p(g)$. So negative formulas don't need to be preserved.
- Now notice $\varphi(y) = \forall x.xy = x$ is only satisfied by e. Since this is a positive formula in the language of *semigroups*, it is preserved by semigroup homs. So if a semigroup has an identity, the identity is preserved under semigroup homs.
- Similarly, $\varphi(x, y) = xyx = x$ is true if and only if $y = x^{-1}$. Since this is positive, it is also preserved by semigroup homs, and so inverses are preserved.
- Mystery solved!

What if we *want* to preserve negative relations too?

Definition

An *embedding* $h : \mathfrak{M} \to \mathfrak{N}$ is a homomorphism that satisfies the following two bonus properties:

- h is injective (that is, $x = y \iff h(x) = h(y)$)
- $r^{\mathfrak{M}}(a_1, \ldots, a_n) \iff r^{\mathfrak{N}}(h(a_1), \ldots, h(a_n))$

Homomorphisms always *preserve* the relations. But an embedding *reflects* the relations too.

Theorem

Embeddings preserve all quantifier free formulas

Proof.

Another easy induction on the definition of formulas. □

But what if we want more? Embeddings aren't enough. If $Z(G)$ is the center of a group G, then Z embeds into G. Yet $Z \models \forall x. \forall y. xy = yx$. If G isn't abelian, this formula is *not* preserved by the embedding.

Intuitively this is because we are quantifying over "extra stuff" when we interpret the quantifiers as ranging over G instead of just $Z(G)$.

Thankfully, we can define our way out of this situation.

Definition

An embedding $h : \mathfrak{M} \to \mathfrak{N}$ is called *elementary* if it preserves and reflects the truth of *all* formulas. That is, for all $a_1, \ldots, a_n \in M$:

$$\mathfrak{M} \models \varphi(a_1, \ldots, a_n) \iff \mathfrak{N} \models \varphi(h(a_1), \ldots, h(a_n))$$

This is obviously a very strong condition. It gives rise to two natural questions:

1. Is there any way to tell if an embedding is elementary?
2. How easy is it to find or construct elementary embeddings?

Answer to Question 1 (Tarski-Vaught)

$h : \mathfrak{M} \to \mathfrak{N}$ is an elementary embedding if and only if whenever $\mathfrak{N} \models \exists y \varphi(h(a_1), \ldots, (a_n), y)$, $\mathfrak{M} \models \exists y \varphi(a_1, \ldots, a_n, y)$ too.

Intuitively, this says that any formula which $\mathfrak{N}$ can make true can also be made true in $\mathfrak{M}$.

Answer to Question 2 (Lowenheim-Skolem)

Let $\mathfrak{M}$ be a model and let $A \subseteq M$. Then A is contained in an elementary submodel $\mathfrak{A}$ of cardinality $\max(|A|, |\mathcal{L}(\sigma)|)$.
In particular, if $\mathcal{L}(\sigma)$ is countable, then every countable $A \subseteq M$ is contained in a countable elementary submodel of $\mathfrak{M}$.

Thank you!