2021 Algebra Qual, Part A

Solve 4 problems. Please clearly indicate the 4 problems to grade.

1. Find the number of elements of order 11 in a simple group of order $748 = 2^2 \cdot 11 \cdot 17$.

2. Let G be a finite group and p be a prime number. Let P be a Sylow p-subgroup of G and let H be a p-subgroup of G such that $H \subseteq N_G(P)$, where $N_G(P)$ denotes the normalizer of P in G. Show that $H \subseteq P$.

3. Let G be a finite simple group of even order. Prove that G is generated by elements of order 2.

4. Let G be a group and let Z denote its center. Suppose that G/Z is cyclic. Prove that G is abelian.

5. Let $D = \mathbb{Z}[\,i\sqrt{5}\,] = \{a + b\,i\sqrt{5} \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$, where $i^2 = -1$. Show that the ring D is not a principal ideal domain.

2021 Algebra Qual, Part B

Do any 4 problems.

1. Recall that if k is a field, then the ring $M_{n\times n}(k)$ of $n \times n$ matrices with entries in k has no nontrivial 2-sided ideals.
   Prove that there exists a nontrivial 2-sided ideal of $M_{n\times n}(\mathbb{Z})$.

2. Let $R = \mathbb{C}[x]$, and let M be an R-module which is finite dimensional as a vector space over $\mathbb{C}$. Suppose there exists a nonzero vector $m \in M$ which is not an eigenvector for x. Prove the restriction map

$$\mathrm{Hom}_R(M, M) \to \mathrm{Hom}_{\mathbb{C}}(M, M)$$

   is *not* surjective.

3. Let R be a unital ring. Show that the following two conditions are equivalent:
   a) Every unital R-module is projective.
   b) Every unital R-module is injective.

4. Let R be a unital commutative ring, and let M, N, P be R-modules. Prove the following R-modules are isomorphic:

$$\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)) \quad \text{and} \quad \mathrm{Hom}_R(N, \mathrm{Hom}_R(M, P))$$

5. Let $R = \mathbb{C}[x]$ and $M = R/(f_c)$, where $c \in \mathbb{C}$ and $f_c(x) := x^2 + c$. Prove that the following two statements are equivalent:
   a) There exists an isomorphism $M \cong M' \oplus M''$, where M' and M'' are nontrivial R-modules.
   b) $c \neq 0$.

2021 Algebra Qual, Part C

Do any 4 problems. Please clearly indicate the 4 problems to grade.

1. Let K be a field. Let $F = K(x_1, \ldots, x_n)$, the field of rational functions in n indeterminates $x_1, \ldots, x_n$ over K. Let $\sigma : F \to F$ be a K-homomorphism. Let $E = \sigma(F)$. Prove that F/E is a finite extension.

2. Is $\mathbf{Q}(i\sqrt{3},\, 5^{1/3})/\mathbf{Q}$ a Galois extension? Justify your answer.

3. Let F/K be a normal algebraic extension. Let $f(x) \in K[x]$ be an irreducible polynomial. Suppose that $u, v \in F$ are two roots of $f(x)$. Prove that there exists a K-automorphism $\sigma : F \to F$ such that $\sigma(u) = v$.

4. Let F be a subfield of $\mathbf{C}$ such that $F/\mathbf{Q}$ is a finite Galois extension whose Galois group is isomorphic to A_5. Prove that $F \cap \mathbf{Q}(e^{2\pi i/n}) = \mathbf{Q}$ for every integer $n \geqslant 1$. (You may use the fact that A_5 is a simple group.)

5. Let K be a finite field. Let $f(x) \in K[x]$ be a monic irreducible polynomial. Prove that $f(x)$ divides $x^{q^n} - x$, where $q = |K|$ and $n = \deg f(x)$.