Chris Grossack

$\boxed{A}$

[redacted] 39/40

① find the # of elems of order 11 in a simple group of order $2^2 \cdot 11 \cdot 17$

[redacted]

9/10

we know the number of 11-sylow subgroups is 1 mod 11 and divides $2^2 \cdot \cancel{11} \cdot 17$, [redacted] Ok

and each element of order 11 lives in a (unique!) such subgroup. (since any two ^prime order cyclic groups of prime order −1 that intersect nontrivially are the same) order is exactly 11

[redacted] (it is below)

Now the only valid choices are 1 or 34 subgroups, but it cannot be 1, as G is simple (so the 11-sylow subgroup is not normal)

So there are 34 11-sylow subgroups, each contributing exactly 10 elements of order 11. This gives 340 total.

No repetitions ✓ [redacted]

② Let $\underline{P}$ be a p-sylow subgroup of G,

10/10

$H \leq N_G \underline{P}$ a p-group. Show $H \subseteq \underline{P}$.

⌜ since $\underline{P} \trianglelefteq N_G \underline{P}$, we know $\underline{P}$ is the <u>unique</u> p-Sylow subgroup of $N_G \underline{P}$. ✓

As H is a p-group contained in $N_G \underline{P}$, it must be contained in some Sylow p-group, but the only choice is $\underline{P}$. ✓

So $H \leq \underline{P}$, as desired. ✓

⌟

~~3~~ Chris Grossack

A

③ let G be a simple group of even order. Show G is generated by elements of order 2.

10/10

⌜let $H = \langle x_i \mid x^2 = 1 \rangle \leq G$. ✓

since $|G|$ is even, it has an element of order 2, so $H \neq 1$. ✓

Now, note $H \trianglelefteq G$, since each generator conjugates to another generator:

for any g, $x^2 = 1$:

$$(g^{-1} x g)^2 = g^{-1} x^2 g = g^{-1} g = 1$$

So $H \neq 1$ is normal in G, so by simplicity $H = G$. ✓

⌟

④ let Z be the center of G.

10/10 if G/Z is cyclic, then G is abelian.

⌜ let $x, y \in G$. Say G/Z is generated by gZ.

then $xZ = (gZ)^k = g^k Z$, $yZ = (gZ)^l = g^l Z$

in G/Z. So $x = g^k z_1$, $y = g^l z_2$ (in G)

for some $z_1, z_2 \in Z$. ✓

Now $xy = g^k z_1 g^l z_2 = g^l z_2 g^k z_1 = yx$ ✓

since z_1, z_2 commute with everything,

and g^k, g^l commute with eachother. ✓

⌟

Chris Grossack

10, 10, 10, 10, N → 40

$\boxed{B}$

10

① Show $M_{n\times n}(\mathbb{Z})$ has a nontrivial (2-sided) ideal

⌈ look at $E \triangleq \{A \in M_{n\times n}(2\mathbb{Z})\}$, the matrices with even entries.

this is clearly closed under +, and contains 0.

Moreover, if A is any matrix and $M \in E$:

$(AM)_{ij} = \sum_k A_{ik} M_{kj}$ is a sum of even numbers (since each M_{kj} is even).

Showing $(MA)_{ij}$ is even is similar,

and we see E is a 2-sided ideal.

(oh, it's nontrivial because it's clearly nonzero (put a 2 in one entry) and not everything (doesn't contain the identity matrix))

⊥

② Let $R = \mathbb{C}[x]$, M an R-mod that's a f-dim $\mathbb{C}$-VS. Say $m \in M$ is not an eigenvector for x. Then $\mathrm{Hom}_R(M,M) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(M,M$ is not surjective.

⌜extend $\{m, xm\}$ to a basis $\{m, xm, v_1 \cdots v_k\}$ of M as a $\mathbb{C}$-vector space. (note m, xm are independent since m is not an eigenvector)

Let $L : M \to M$ be the map of $\mathbb{C}$-VSs linearly extending

$$m \longmapsto m$$
$$v_i \longmapsto 0 \quad \forall i.$$
$$xm \longmapsto 0$$

now $L \in \mathrm{Hom}_{\mathbb{C}}(M, M)$, but L is not the restriction of an R-linear map, since if it were, we would have:

10

$$0 = L(xm) = x\,Lm = xm \neq 0 \qquad \times$$

(again, note $xm \neq 0$ since otherwise m would be an eigenvector w/ eigenvalue 0)

⌟

Chris Grossack

$\boxed{B}$

③ let R be a (unital) ring. TFAE

$\boxed{I}$ every (unital) R-mod is projective

$\boxed{II}$ " " " " injective

⌜ We show $\boxed{I} \Leftrightarrow$ "every short exact sequence splits" $\Leftrightarrow \boxed{II}$

if every C is projective, then every $0 \to A \to B \to C \to 0$ splits by considering

$$\begin{tikzcd} & C \arrow[dl, dashed] \arrow[d, "id"] \\ B \arrow[r, two heads] & C \end{tikzcd}$$

Conversely if every SES splits, then every P is projective:

$$\begin{tikzcd} & P \arrow[dl, dashed, "sf"'] \arrow[d, "f"] \\ B \arrow[r, "\pi"] & C \arrow[l, bend left, "s"] \end{tikzcd}$$

(where s splits $0 \to \ker\pi \to B \underset{s}{\overset{\pi}{\rightleftarrows}} C \to 0$) ✓

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So $\boxed{I} \Leftrightarrow$ "every SES splits",

Dually, if every A is injective, then every $0 \to A \to B \to C \to 0$ splits by considering

$$\begin{tikzcd} A \\ A \arrow[u, "id"] \arrow[r, hook] & B \arrow[ul, dashed] \end{tikzcd}$$

and if every SES splits then every I is injective:

$$\begin{tikzcd} I \\ A \arrow[u, "f"] \arrow[r, hook, "i"] & B \arrow[ul, dashed, "fs"'] \arrow[l, bend left, "s"] \end{tikzcd}$$

(where now s splits $0 \to A \underset{s}{\overset{i}{\rightleftarrows}} B \twoheadrightarrow \operatorname{coker} i \to 0$)

So "every SES splits" $\Leftrightarrow \boxed{II}$, and we're done.

⌟

④ $\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)) \cong \mathrm{Hom}_R(N, \mathrm{Hom}_R(M, P))$ ....

⌈ By yoneda, it suffices to show (naturally in X)

$$\mathrm{Hom}\left(X, \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))\right) \cong \mathrm{Hom}\left(X, \mathrm{Hom}_R(N, \mathrm{Hom}_R(M, P))\right)$$

but by the tensor hom adjunction and commutativity of $\otimes$, we have:

$$\begin{aligned}\mathrm{Hom}(X, \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))) &\cong \mathrm{Hom}(X \otimes M, \mathrm{Hom}_R(N, P)) \\ &\cong \mathrm{Hom}(X \otimes M \otimes N, P) \\ &\cong \mathrm{Hom}(X \otimes N \otimes M, P) \\ &\cong \mathrm{Hom}(X \otimes N, \mathrm{Hom}_R(M, P)) \\ &\cong \mathrm{Hom}(X, \mathrm{Hom}_R(N, \mathrm{Hom}_R(M, P)))\end{aligned}$$

⌋

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(could just do $\mathrm{Hom}(M, \mathrm{Hom}(N, P))$

$$\begin{aligned}&\cong \mathrm{Hom}(M \otimes N, P) \\ &\cong \mathrm{Hom}(N \otimes M, P) \\ &\cong \mathrm{Hom}(N, \mathrm{Hom}(M, P))\end{aligned}$$
)

Chris Grossack

$\boxed{C}$

36/40

② Is $\mathbb{Q}(i\sqrt{3}, \sqrt[3]{5})$ galois?

yes. since $\mathbb{Q}$ is characteristic 0, it is separable. For normal, we show every conjugate root is also in $\mathbb{Q}(i\sqrt{3}, \sqrt[3]{5})$.

$i\sqrt{3}$ is a root of the quadratic x^2+3.
So $-i\sqrt{3}$, the other root, is also in the field.

$\sqrt[3]{5}$ is a root of x^3-5, and the other roots are $\sqrt[3]{5}\,\omega$, $\sqrt[3]{5}\,\omega^2$ for ω a primitive cube root of unity.

Thankfully, we can take $\omega = \dfrac{-1+\sqrt{3}i}{2}$, which is in the field, so the conjugate roots of $\sqrt[3]{5}$ are in the field too, and it is normal. Thus, galois.

✓ ⑩

③ let F/k normal, algebraic. let $f \in k[x]$ irreducible. If u, v are two roots of f in F, show there is a k-automorphism $\sigma : F \to F$ with $\sigma(u) = v$.

⌜We build $\tilde{\sigma} : F \to \bar{k}$ as

$$\begin{tikzcd}
F \arrow[r, dashed, "\tilde{\sigma}"] & \bar{k} \\
k(u) \arrow[u] \arrow[r, "u \mapsto v"] & k(v) \arrow[u] \\
k \arrow[u] \arrow[r] & k \arrow[u]
\end{tikzcd}$$

by using u, v conjugate here → $k(u)$

and the nice extension theorems we have for maps of fields.

But since F/k is <u>normal</u>, any $\tilde{\sigma} : F \to \bar{k}$ restricts to an automorphism $\sigma : F \to F$. As needed. ⌟

✓ ⑩

Chris Grossack

$\boxed{C}$ ④ Let F be a subfield of $\mathbb{C}$ with $F/\mathbb{Q}$ finite, galois whose galois group $G \cong A_5$. Show $F \cap \mathbb{Q}(e^{2\pi i/n}) = \mathbb{Q}$ for every $n \geq 1$.

⌜we know $\mathbb{Q}(e^{2\pi i/n}) = \mathbb{Q}(\zeta_n)$ is galois, since the conjugate roots are ζ_n^k for $1 \leq k \leq n$.

So we have a diagram

$$\begin{tikzcd}
F \arrow[dr, dash] & & \mathbb{Q}(\zeta_n) \arrow[dl, dash] \\
& F \cap \mathbb{Q}(\zeta_n) \arrow[d, dash] & \\
& \mathbb{Q} &
\end{tikzcd}$$

but the intersection of galois extensions is galois, ✓

So we get a tower

$$\begin{tikzcd}
F \arrow[d, dash] \\
F \cap \mathbb{Q}(\zeta_n) \arrow[d, dash] \\
\mathbb{Q}
\end{tikzcd}$$

where $F \cap \mathbb{Q}(\zeta_n) / \mathbb{Q}$ is normal.

Then $\mathrm{gal}\left(\overset{F/(F\cap\mathbb{Q}(\zeta_n))}{\cancel{F \cap \mathbb{Q}(\zeta_n)/\mathbb{Q}}}\right) \lhd G$, and so is either all of A_5 or 1, since $G = A_5$ is simple. ⑧

But $\overset{\text{why?}}{\boxed{F \neq F \cap \mathbb{Q}(\zeta_n)}}$, so $\mathrm{gal}(F \cap \mathbb{Q}(\zeta_n)/\mathbb{Q}) = 1 = \mathrm{gal}(\mathbb{Q}/\mathbb{Q})$

and by the galois correspondence, $F \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$.

⑤ Let K be a finite field, $f \in K[x]$ monic irreducible.

Prove $f \mid x^{q^n} - x$ where $|K| = q$ and $n = \deg f$.

⌈ Let α be a root of f in some algebraic closure.

now $K(\alpha) \cong \frac{K[x]}{f}$ has size q^n (since it's an n-dim vector space over K)

Now the multiplicative group $K(\alpha)^{\times}$ has order $q^n - 1$.

So $\alpha^{q^n - 1} = 1$ by group theory, and $\alpha^{q^n} = \alpha$. ✓

So α is also a root of $x^{q^n} - x$, and since this is true of every root of f we see $f \mid x^{q^n} - x$. ⌋

✓

explain:
use f is min. poly of its roots,
or use f is separable so no repeated roots.

⑧