Real Analysis Qualifier — 2021

Undergraduate Problems

Choose one problem from 1–2 and one from 3–4. Please show all work. Unsupported claims will not receive credit.

1. Prove using the ϵ-δ definition of continuity that $f(x) = x^2$ is continuous on $\mathbb{R}$.

2. Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at x_0 and $f(x_0) > 0$ then $g = \sqrt{f}$ is differentiable at x_0 as well. Note g is well defined near x_0. Do this using only the definition of the derivative. (Don't use the product rule, the chain rule, etc., unles you prove these.)

3. Use integration by parts to show that $\int_0^1 \frac{f^2(x)}{x^2} dx \leq 4 \int_0^1 (f'(x))^2 dx$ for all smooth f on $[0, 1]$ such that $f(0) = 0$.

4. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable with $|f'(x)| \leq M$ for all $x \in \mathbb{R}$, where M is some positive real number. Prove that

$$|f(x) - f(y)| \leq M|x - y| \text{ for all } x, y \in \mathbb{R}.$$

209A

Choose one problem from 1–2 and one from 3–4. Please show all work. Unsupported claims will not receive credit.

1. Given a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for all $A \in \mathcal{M}$. Prove that μ_E is a measure; that is, $(X, \mathcal{M}, \mu_E)$ is a measure space.

2. If $f \colon \mathbb{R} \to \mathbb{R}$ is monotone then it is Borel measurable.

3. Given a measure space $(X, \mathcal{M}, \mu)$, let $f \in L^+(\mathcal{M})$.[1] Define $\lambda(E) := \int_E f \, d\mu$ for all $E \in \mathcal{M}$. Prove that λ is a measure on $\mathcal{M}$ and that for all $g \in L^+$,

$$\int g \, d\lambda = \int fg \, d\mu.$$

4. Let $(X, \mathcal{M}, \mu)$ be a measure space. If $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$ and $\mathbb{1}_{E_n} \to f$ in $L^1(\mathcal{M})$ as $n \to \infty$, then f is (almost everywhere) equal to the characteristic function of a measurable set. [2]

[1] Recall that $L^+(\mathcal{M})$ is the the space of all measurable functions from X to $[0, \infty]$.
[2] $\mathbb{1}_{E_n}$ is the characteristic function of E_n, also often written χ_{E_n}.

209B

Choose one problem from 1–2 and one from 3–4. Please show all work. Unsupported claims will not receive credit.

1. Let

$$g(x) = \begin{cases} e^x - 1, & x < 0; \\ e^x, & 0 \le x < 1; \\ x + 3, & x \ge 1. \end{cases}$$

Let ν be the Borel measure generated by g, i.e. $\nu((a,b]) = g(b) - g(a)$ for all numbers $b \ge a$. Let $\nu = \lambda + f d\mu$ be the Lebesque decomposition, where μ is the Lebsgue measure and λ is singular w.r.t. μ. Determine λ and f explicitly.

2. State respectively the definitions that a function is of bounded variation and absolutely continuous on $[0,1]$. Construct a continuous function on $[0,1]$ which is of bounded variation but is not absolutely continuous.

3. Let $X = [0, 2\pi]$, equipped with Lebesgue measure. (a). Let $f_n = \sin^{21}(nx)$. Prove that f_n converges to 0 weakly in $L^2(X)$ as $n \to \infty$. That is, $\lim_{n\to\infty} \langle g, f_n \rangle = 0$ for all $g \in L^2(X)$ where $\langle \cdot, \cdot \rangle$ is the inner product on $L^2(X)$. *You may assume Riemann-Lebesgue Lemma.* (b). Prove that f_n does not converge to 0 a.e.

4. Let $[0,1]$ be equipped with the Lebesgue measure. Let $L^1[0,1]$ be the space of integrable functions and $L^p[0,1]$, $p > 1$, be the space of functions whose p-th power is integrable on $[0,1]$. Show that $L^p[0,1]$ is a meager subset of $L^1[0,1]$, i.e., $L^p[0,1]$ can be written as a countable union of nowhere dense subsets of $L^1[0,1]$. *Hint:* $L^p[0,1] = \bigcup_{N=1}^{\infty} \{f : \|f\|_p \le N\}$.

209C

Choose one problem from 1–2 and one from 3–4. Please show all work. Unsupported claims will not receive credit.

1. Let m be Lebesgue measure on the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Consider the space $(L^\infty(\mathbb{T}, m), \|\cdot\|_{L^1})$ where $\|f\|_{L^1} = \int_{\mathbb{T}} |f| dm$.
   (a) Is it a Banach space? Prove your answer.
   (b) Is it a separable space? Prove you answer. *Here you may use the fact that trigonometric polynomials are dense in $(L^p(\mathbb{T}, m), \|\cdot\|_{L^p})$ for $1 \le p < \infty$.*

2. Let $(X, \mathcal{M}, \mu)$ be measure space where μ is a positive measure. Let $\{f_n\}_{n\ge1}$ be a Cauchy sequence in $(L^1(X,\mu), \|\cdot\|_{L^1})$. Find a subsequence $\{f_{n_k}\}_{k\ge1}$ of $\{f_n\}$ such that the pointwise limit
$$f(x) := \lim_{k\to\infty} f_{n_k}(x)$$
exists for μ-almost every $x \in X$ and $f \in L^1(X, \mu)$.

3. Recall the Schwartz space $\mathcal{S}(\mathbb{R})$ is a Fréchet space via the family of semi-norms
$$\|f\|_k := \sum_{n,m:|n|+|m|\le k} \sup_{x\in\mathbb{R}} \left| x^n f^{(m)}(x) \right|.$$
Consider the linear map $\Lambda : \mathcal{S}(\mathbb{R}) \to \mathbb{R}$ where $\Lambda(f) = \int_{\mathbb{R}} |xf| dx$. Show that Λ is continuous with respect to the natural topology on $\mathcal{S}(\mathbb{R})$.

4. Recall the Fourier Transformation on $\mathcal{S}(\mathbb{R})$ is defined as
$$\hat{f}(\xi) := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} f(x) e^{-ix\xi} dx.$$
Show that $\hat{f} \in C^\infty(\mathbb{R})$ for each $f \in \mathcal{S}(\mathbb{R})$.