Chris Grossack

Undergrad Problems

① Show x^2 is cts on $\mathbb{R}$

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⌈ let $\varepsilon > 0$, $x_0 \in \mathbb{R}$ if $|x_0 - y| < \varepsilon$, then

$$|x_0^2 - y^2| = |x_0 - y|\,|x - y| \leq (2|x_0| + \varepsilon)\varepsilon \longrightarrow 0 \quad \text{as } \varepsilon \to 0.$$

So x^2 is cts @ x_0 for each x_0, and is cts on $\mathbb{R}$. ⌋

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④ if $|f'| \leq M$, show $|f(x) - f(y)| \leq M|x - y|$.

⌈ wlog $x < y$. By the mean value thm,

$$|f(x) - f(y)| = |f'(\xi)|\,|x - y| \quad \text{for some } \xi \in (x, y).$$

now $|f'(\xi)| \leq M$, so $|f(x) - f(y)| \leq M|x - y|$. ⌋

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(1) if $(X, \mathcal{M}, \mu)$ is a measure space, and $E \in \mathcal{M}$,

show $\mu_E A \triangleq \mu(E \cap A)$ is a measure on $(X, \mathcal{M})$ too.

$\ulcorner$ $\mu_E : \mathcal{M} \to [0, \infty]$ since μ is.

now $\mu_E \emptyset = \mu(E \cap \emptyset) = \mu \emptyset = 0$

and if (A_n) is a countable sequence of disjoint measurable sets,

$$\mu_E(\bigcup A_n) = \mu(E \cap \bigcup A_n) = \mu(\bigcup E \cap A_n) = \sum \mu(E \cap A_n) = \sum \mu_E A_n$$

(since the $E \cap A_n$ are also disjoint, measurable).

$\lrcorner$

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A ③ let $f \in L^+$. 10/10 write $\lambda E \triangleq \int_E f \, d\mu$ for $E \in \mathcal{M}$.
Prove λ is a measure on $(X, \mathcal{M})$ and
$\int g \, d\lambda = \int g f \, d\mu$ for each $g \in L^+$.

⌈ since $f \geq 0$, $\lambda : \mathcal{M} \to [0, \infty]$.
now $\lambda \phi = \int_\phi f \, d\mu = 0$
and if (A_n) is a disjoint sequence of sets in $\mathcal{M}$, then

$$\lambda(\cup A_n) = \int \left(\sum_{n=1}^{\infty} \mathbb{1}_{A_n}\right) \cdot f \, d\mu \overset{MCT}{=} \lim_{N \to \infty} \int \left(\sum_{n=1}^{N} \mathbb{1}_{A_n}\right) \cdot f \, d\mu$$

$$= \lim_{N \to \infty} \sum_{n=1}^{N} \int_{A_n} f \, d\mu = \sum_{n=1}^{\infty} \lambda A_n.$$

Now let $g \in L^+$.
if g is simple, say $g = \sum_{n=1}^{N} a_n \mathbb{1}_{E_n}$, then

$$\int g \, d\lambda = \sum_{n=1}^{N} a_n \lambda E_n = \sum_{n=1}^{N} a_n \int \mathbb{1}_{E_n} \cdot f \, d\mu = \int \left(\sum_{n=1}^{N} a_n \mathbb{1}_{E_n}\right) f \, d\mu = \int g f \, d\mu.$$

if g is arbitrary, fix an increasing sequence of simple functions $\phi_n \nearrow g$.
By the MCT (twice!)

$$\int g \, d\lambda = \int \lim \phi_n \, d\lambda \overset{MCT}{=} \lim \int \phi_n \, d\lambda = \lim \int \phi_n f \, d\mu \overset{MCT}{=} \int \lim \phi_n f \, d\mu$$

$$= \int g f \, d\mu.$$

as desired. ⌋

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$\boxed{B}$ 1 let $g(x) = \begin{cases} e^x - 1 & x < 0 \\ e^x & 0 \leq x < 1 \\ x+3 & 1 \leq x \end{cases}$

+6.

if ν is the borel measure given by g,

find $\nu = \lambda + f\,d\mu$ the lebesgue decomposition.

$\ulcorner$ $f = \frac{d\nu}{d\mu} = g'$ wherever g is cts (thus differentiable, since g is monotone)

so $f(x) = \begin{cases} e^x & x < 0 \\ e^x & 0 < x < 1 \\ 1 & 1 < x \end{cases}$

this is enough since f is only defined up to nullsets anyways.

Now λ is given by ~~the~~ the points of discontinuity:

$$\lambda = \delta_0 + (4-e)\,\delta_1$$

Need to check these two measure are equal on each interval $[a, b)$. $b<0$, $a<b<1$, $b>1$ ...

where δ_{x_0} is the dirac measure @ x_0.

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$\boxed{B}$ (4) Show $L^p[0,1]$ is meagre in $L^1[0,1]$ for $p > 1$.

+/0

Since $[0,1]$ has finite measure, $L^p \subseteq L^1$.

Now let $i : L^p \hookrightarrow L^1$ be the inclusion.

Note both spaces are Banach and i is bounded, since $\|if\|_1 \leqslant \|1\|_q \|f\|_p$ by Hölder's inequality.

$\Downarrow$ if the image $i[L^p]$ were nonmeagre, by the generalized ~~open mapping thm,~~ it would need to be surjective.

But $x^{-1/p}$ is in L^1 ✓ but not L^p, since $\int_0^1 x^{-1/p} = \frac{1}{1-\frac{1}{p}}$, $\int_0^1 (x^{-1/p})^p = \int_0^1 x^{-1} = \infty$.

So i is not surjective, thus must have meagre image in L^1.

$\lrcorner$

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$\boxed{C}$ (1) a) is $(L^\infty(\mathbb{T}), \|\cdot\|_1)$ a Banach Space?

⌈ no. if f is any positive unbounded L^1 function, (say $x^{-1/2}$), then f is the L^1 limit of the sequence $f_n = \min(f, n)$ of L^∞ functions ($\|f_n\|_\infty \leq n$). Now $f_n \xrightarrow{L^1} f$, but f is not L^∞! So $(L^\infty, \|\cdot\|_1)$ is not Banach.

Good

b) is it Separable?

⌈ yes L^1 is well known to be Separable, (consider trig polynomials w/ rational coefficients) and any uncountable discrete set in $(L^\infty, \|\cdot\|_1)$ would be uncountable discrete in $(L^1, \|\cdot\|_1)$ too.

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$\boxed{C}$ (4) $\hat{f}$ is C^∞ for each $f \in \mathcal{S}(\mathbb{R})$.

not exactly right

⌈ recall if $x^k f$ is L^1, then $\hat{f}$ is C^k.

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Indeed

$$\frac{d^k}{d\xi^k} \hat{f}(\xi) = \frac{d^k}{d\xi^k} \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ix\xi} dx$$

Apply DCT

$$\left(\begin{array}{c}\text{Leibniz,}\\ \text{Since}\\ e^{ix\xi} \text{ is } C^\infty\\ \text{in } \xi\end{array}\right) = \frac{1}{\sqrt{2\pi}} \int (-ix)^k f(x) e^{-ix\xi} dx$$

$$= (-i)^k \widehat{x^k f}.$$

not by def, need a proof

Now, if $f \in \mathcal{S}(\mathbb{R})$, each $x^k f$ is L^1 by definition.

So $\hat{f}$ is C^k for all k, and $\hat{f} \in C^\infty$. ⌟