Syntax and Semantics

A Guiding Principle

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(they/them)

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- Since we only have 5 minutes together, I'm going to go rather fast
- I stuck to simple examples, but there's many many more
- You can find these slides on my blog at grossack.site
- I'm also going to write up a blog post in the near future which goes into more depth, and lists some other examples too.
- Let's start!

The observation that makes the whole field of logic work is this:

Symbols have no meaning!

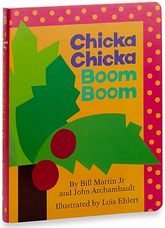
The Syntax of mathematics is the symbols we use to write down our ideas.

The Semantics of mathematics is the *interpretation* of those symbols that we as humans use to endow the symbols with meaning.

By placing restrictions on what we are allowed to say *syntactically* we gain information about what our objects can be *semantically*. Often without needing to reason about the objects themselves!

An Instructive Example

Which object is more complicated?



You don't need to have read either to know!

The *length* of the book (a purely syntactic notion) gives a bound on how complicated the book can be (a semantic notion)

One for the Algebraists

Definition

Say $G = \langle a_1 \dots a_n \mid R_1 \dots R_m \rangle$ is the presentation of a group.
This means the elemnts of G are all the words you can write down if you use $a_1 \dots a_n$ as your alphabet.
Multiplication is done by putting two words next to each other
Two words are considered "equal" if you can get from one to the other by adding or removing the words $R_1 \dots R_m$.

One for the Algebraists

e.g.

If $G = \langle a, b \mid a^{-1}b^{-1}ab \rangle$, then
Elements are

- $aba^{-1}aab$
- $bbaab^{-1}b^{-1}a$
-
- etc.

Multiplication is "Concatenate 'n' Reduce"

- $(aba) \cdot (bba^{-1}) = ababba^{-1}$
- $(ab) \cdot (b^{-1}) = a$
- $(ba^{-1}b^{-1}) \cdot (ab) = ba^{-1}b^{-1}ab = b$
- etc.

This describes the Syntax of a group G. Semantically, one can show $G \cong \mathbb{Z}^2$.

One for the Algebraists

What happens when we impose a *constraint* on the syntax?

Only one letter allowed?	cyclic!
Only one relation allowed?	very rich structure!
$\{w \mid w = 1\}$ is "simple"?	depends what you mean by "simple"

- $\{w \mid w = 1\}$ is Regular $\iff$ G is finite
- $\{w \mid w = 1\}$ is Context Free $\iff$ G is virtually free

One for the Algebraists

- In general "free" objects are pure syntax
- Since every algebraic object can be wrtten as a quotient of a free object, you can *always* ask these kinds of questions.
- For example, if the ideal I of a polynomial ring R is generated by only monomials (a syntactic condition), what can you say about R/I (semantically)?

One for the Analysts

Meaasurability

A subset $E \subseteq \mathbb{R}^n$ is called Measurable if you can consistently assign it a volume.
Meaasurability is a kind of regularity – it says E cannot be too complicated.

Baire

A subset $E \subseteq \mathbb{R}^n$ has the Baire Property if it is "almost open" (it differs from an open set by a meagre set).
This is another kind of regularity. Sets with the Baire Property are nearly open, and so they cannot be too complicated.

One for the Analysts

Theorem

If $\{(\overline{x}, \overline{y}) \mid \phi(\overline{x}, \overline{y})\}$ is Borel, then

- $E = \{\overline{x} \mid \exists \overline{y}.\phi(\overline{x}, \overline{y})\}$ is Universally Measurable and has the Baire Property
- $E = \{\overline{x} \mid \forall \overline{y}.\phi(\overline{x}, \overline{y})\}$ is Universally Measurable and has the Baire Property

Defnition

Sets of this form are called Analytic and CoAnalytic respectively.

One for the Analysts

- What happens when we add more quantifiers?
- These sets are allowed to be less simple, but how simple *are* they?
- This is a central question in descriptive set theory
- It turns out the answer depends on what axioms you choose!

- These sorts of theorems are all around us!
- Be on the look out for syntactic restrictions on the objects you see
- If you're *studying* a family of objects, one natural question might be "what happens if I restrict the syntax in some way?"
- This idea runs deep – The P $\stackrel{?}{=}$ NP problem exactly asks whether programs "with an existential quantifier" are as simple as those without.

Thank You!