Where Are The Open Sets?

Comparing HoTT with classical Topology.

Chris Grossack (they/them)

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Some Disclaimers.

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↳ (but I'll always link to more comprehensive references)

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↳ how do we know that our theorems are true for a classical topologist?

↳ HoTT is a <u>logic</u>, so we want to get <u>semantics</u>.

HoTT

$$\text{HoTT} \xrightarrow{\text{semantics}} \text{Simplicial sets}$$

$$
\begin{tikzcd}
\text{HoTT} \arrow[r, squiggly, "\text{semantics}"] & \text{Simplicial Sets} \arrow[d, "\text{Quillen Equivalence}"] \\
& \text{Topological Spaces}
\end{tikzcd}
$$

The Quillen Equivalence says that any category-theoretic property of simplicial sets (sSet) up to homotopy is <u>also true</u> of topological spaces (Top) up to homotopy.

But, through our semantics, HoTT will tell us true facts about sSet!

Let's get started!

§ 1

Model Categories

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(a sketch)

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- $H(-, 1) = g(-)$

Such an H is called a <u>homotopy</u> from f to g.

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written $H: f \simeq g$

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- $f \circ g \sim id_Y$
- $g \circ f \sim id_X$

This looks like some kind of isomorphism...

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If only we could make

$\sim$ into equality...

This looks like some kind of isomorphism...

If only we could make $\sim$ into equality...

But what if we simply __forced__ that to be true?

Def$^{\underline{n}}$

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- $\mathcal{W}$ is the class of (Weak) Homotopy Equivalences in Top

$\underline{\text{Def}^{\underline{n}}}$

- $\mathcal{W}$ is the class of (Weak) Homotopy Equivalences in Top
- $\text{Top}[\mathcal{W}^{-1}]$ is the category we get by freely adding inverses for each $f \in \mathcal{W}$.

Good news :

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(weak) homotopy equivalence

is now isomorphism!

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(weak) homotopy equivalence
is now isomorphism!

↳ this lets us do category
theory more directly, and
is very useful™.

Bad News:

$\mathrm{Top}[\mathcal{W}^{-1}]$ is <u>awful</u> to work with!

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↳ basically no limits or colimits, for instance.

↳ More importantly, the arrows are <u>extremely</u> complicated.

Good News Again:

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There is a notion of a Model Structure that will let us compute with $Top[w^{-1}]$

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There is a notion of a <u>Model Structure</u> that will let us compute with $\mathrm{Top}[w^{-1}]$

(that is, with spaces up-to-homotopy)

Bad News Again:

I'm not going to tell you what a model structure is.

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(See "Homotopy Theories and Model Categories" by Dwyer & Spalinski)

Briefly, the idea is to choose <u>Fibrations</u>

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Briefly, the idea is to choose Fibrations and Cofibrations in our category, which satisfy certain axioms related to each other and to the weak Equivalences $\mathcal{W}$.

Then we can compute in $\mathrm{Top}[w^{-1}]$ by making computations in Top, restricting attention to the objects that are both <u>fibrant</u> and co<u>fibrant</u>.

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(Again, see Dwyer & Spalinski for details)

But what does this have to do with simplicial sets?

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More importantly... what <u>is</u> a simplicial set?

The idea is to get a <u>combinatorial model</u> for topological spaces.

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↳ glue together

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points   arcs

•   •—•

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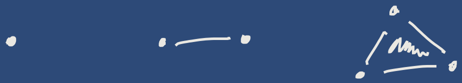
points   arcs   triangles



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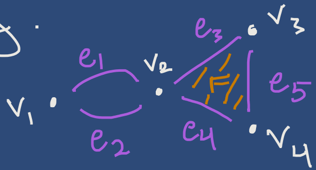

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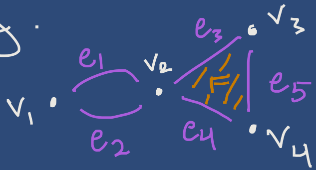
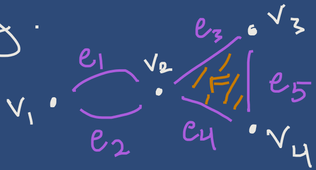


0-cells: $\{v_1, v_2, v_3, v_4\}$

e.g.



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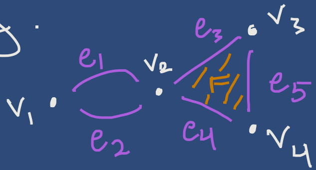

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2-cells: $\{F\}$

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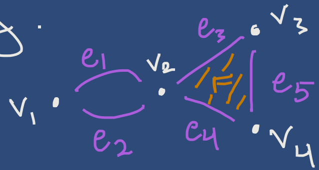

0-cells: $\{v_1, v_2, v_3, v_4\}$

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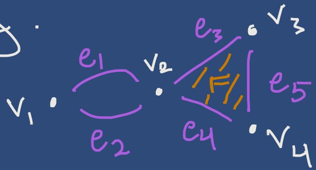
We also have

<u>face maps</u>

eg.



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eg:

- $d_1 e_1 = v_2$
- $d_0 e_1 = v_1$

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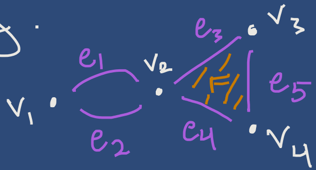

0-cells: $\{v_1, v_2, v_3, v_4\}$

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We also have face maps

eg:

- $d_1 e_1 = v_2$
- $d_0 e_1 = v_1$

- $d_0 F = e_3$
- $d_1 F = e_4$
- $d_2 F = e_5$

In general, we package this information into a diagram

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X_0

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$$X_0 \leftleftarrows X_1$$

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$$\begin{tikzcd} X_0 & X_1 \arrow[l, shift left] \arrow[l, shift right] & X_2 \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \end{tikzcd}$$

In general, we package this information into a diagram

$$X_0 \leftleftarrows X_1 \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; X_2 \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; X_3$$

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$$\begin{tikzcd}
X_0 \arrow[r, shift left=1] & X_1 \arrow[l, shift left=3] \arrow[l, shift right=1] \arrow[r, shift left=2] \arrow[r, shift right=2] & X_2 \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & X_3 \arrow[l, shift left=6] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[l, shift right=6] & \cdots
\end{tikzcd}$$

In general, we package this information into a diagram

$$
\begin{tikzcd}
X_0 \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] & X_1 \arrow[r, leftarrow, shift left=4] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[r, leftarrow, shift right=4] & X_2 \arrow[r, leftarrow, shift left=6] \arrow[r, shift left=4] \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, shift right=4] \arrow[r, leftarrow, shift right=6] & X_3 \quad \cdots
\end{tikzcd}
$$

Conveniently realized as a functor into Sets!

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$$
\begin{tikzcd}
X_0 \arrow[r, shift left=2] & X_1 \arrow[l, shift left=4] \arrow[l] \arrow[r, shift left=2] \arrow[r, shift left=6] & X_2 \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] \arrow[r, shift left=2] \arrow[r, shift left=6] \arrow[r, shift right=2] & X_3 \arrow[l, shift left=4] \arrow[l, shift left=8] \arrow[l] \arrow[l, shift right=4] & \cdots
\end{tikzcd}
$$

Conveniently realized as a functor into Sets!

We call this functor category $\underline{\mathrm{sSet}}$.

For more information, see

↳ Friedman's
"An Elementary Illustrated Introduction to Simplicial Sets"

↳ Singh's
"A Survey of Simplicial Sets"

Ok... But why are we talking about this?

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It turns out there is a model structure on sSet too!

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(Fibrant objects are called <u>Kan Complexes</u>)

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↳ these play nicely with the model structures on sSet and Top.

Thm

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$\boxed{\mathrm{I}}$ 1.1 and Sing(·) induce functors

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- $\mathbb{L}|\cdot| : sSet[w^{-1}] \longrightarrow Top[w^{-1}]$

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- $\mathbb{L}|\cdot| : sSet[w^{-1}] \longrightarrow Top[w^{-1}]$
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Thm

$\boxed{\mathrm{I}}$ $|\cdot|$ and $\mathrm{Sing}(\cdot)$ induce functors

- $\mathbb{L}|\cdot| : sSet[w^{-1}] \longrightarrow Top[w^{-1}]$
- $\mathbb{R}\mathrm{Sing} : Top[w^{-1}] \longrightarrow sSet[w^{-1}]$

$\boxed{\mathrm{II}}$ these form an adjoint equivalence

$$Top[w^{-1}] \simeq sSet[w^{-1}]$$

(this is discussed in more detail in the previously mentioned sources)

The punchline is that <u>any</u> question we have about topological spaces up to homotopy

(as long as it is expressible in the language of category theory)

can be answered by looking at $\mathbb{R}\mathrm{Sing}$ of all the spaces involved, and working in sSet instead!

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↳ ΔSet is "combinatorial", so it's easier to describe to a computer (i.e. Sage)

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↳ sSet is "combinatorial", so it's easier to describe to a computer (i.e. Sage)

↳ sSet is a Topos, giving us access to lots of heavy duty category theoretic machinery.

§ 2

What does this have to do with HoTT?

Lots!

Lots!

But be warned:

this is not for

the faint of heart...

I don't have time to give much detail.

For more, see

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↳ Riehl's "On the ∞-topos semantics of HoTT" (lectures 1, 2, 3 all on youtube)

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↳ Kapulkin & Lumsdaine's "The Simplicial Model of Univalent Foundations"

Morally, the idea is this:

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$\hookrightarrow$ types in HoTT get interpreted as Kan complexes

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↳ types in HoTT get interpreted as Kan complexes

↳ Constructions on types get interpreted as constructions on Kan complexes.

eg.

eg.

- dependent types $x:A \vdash B(x)$ type get modeled as fibrations

$$\begin{array}{c} [\![B]\!] \\ \downarrow \\ [\![A]\!] \end{array}$$

eg.

- then Σ and Π types are given by adjunctions to pullback $\Sigma \dashv pr^* \dashv \Pi$

e.g.

- Identity types are interpreted as path spaces

$$\begin{tikzcd} A \arrow[r, tail, "\mathrm{refl}", "\sim"'] \arrow[dr, "\Delta"'] & {[\![x = y]\!]} \arrow[d, two heads] \\ & A \times A \end{tikzcd}$$

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- Identity types are interpreted as path spaces

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more concretely,

$$\begin{tikzcd} {[\![x=y]\!]} \arrow[d, two heads] \\ A\times A \end{tikzcd} \quad \text{is} \quad \begin{tikzcd} A^{\cdot \to \cdot} \arrow[d, two heads] \\ A\times A \end{tikzcd}$$

eg.

This "explains" path induction!

$$\begin{tikzcd}
A \arrow[r, "p_0"] \arrow[d, "\sim"', "\mathrm{refl}"] & \underline{P} \arrow[d, two heads] \\
A^{\bullet \to \bullet} \arrow[ur, dashed, "J"] \arrow[r, equal] & A^{\bullet \to \bullet}
\end{tikzcd}$$

eg.

This "explains" path induction!

$$\begin{tikzcd}
A \arrow[r, "p_0"] \arrow[d, "s"', "\mathrm{refl}"] & P \arrow[d] \\
A^{\bullet \to \bullet} \arrow[ur, dashed, "J"] \arrow[r, equal] & A^{\bullet \to \bullet}
\end{tikzcd}$$

- $p_0 : P(a, a, \mathrm{refl}_a)$

  says the square commutes

eg.

This "explains" path induction!

$$\begin{tikzcd} A \arrow[r, "p_0"] \arrow[d, tail, "s"', "\mathrm{refl}"] & P \arrow[d, two heads] \\ A^{\cdot \to \cdot} \arrow[r, equal] \arrow[ur, dashed, "J"] & A^{\cdot \to \cdot} \end{tikzcd}$$

$J(x,y,\alpha) : P(x,y,\alpha)$

Says the bottom triangle commutes

eg.

This "explains" path induction!

$$\begin{tikzcd}
A \arrow[r, "p_0"] \arrow[d, "s"', "\mathrm{refl}"] & P \arrow[d] \\
A^{\cdot\to\cdot} \arrow[ur, dashed, "J"] \arrow[r, equal] & A^{\cdot\to\cdot}
\end{tikzcd}$$

$J(a,a,\mathrm{refl}_a) \doteq p_0(a)$

Says the top triangle commutes.

Unfortunately, everything I just told you is a lie.

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↳ in HoTT, we have on-the-nose equality of syntax

↳ in ΔSet, our structure is only defined up-to-isomorphism so we need to evil-ify sSet somehow, and then do all these constructions in that setting.

so we need to evil-ify sSet somehow, and then do all these constructions in that setting.

↳ the key is called a (weakly) <u>universal fibration</u>

From a universal fibration $\mathcal{U}$ we can turn every construction into a map from (some variant of) $\mathcal{U}$ to itself.

Then each construction is a pullback along a variant, and by choosing a particular pullback in each case, we get equality on-the-nose.

(see the incredibly lucid explanation in Kapulkin & Lumsdaine's "The Simplicial Model of Univalent Foundations" for more)

If you decide to go through with this, here's a good exercise to keep in mind:

If $\mathrm{isContr}(A)$ is inhabited, show the Kan complex $[\![A]\!]$ is contractible.

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(use the ΔSet semantics
of HoTT)

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Next, show the geometric realization $|[\![A]\!]|$ is contractible.

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Next, show the geometric realization $|[\![A]\!]|$ is contractible.

(use the properties of $|\cdot| \dashv \mathrm{Sing}$)

Then pat yourself on the back for proving something about an honest topological space using HoTT!

Thank You

^_^